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CONTENTS

Geophysics. — *The Earth's crust deformation in the East Indies.* (Provisional Paper.) By F. A. VENING MEINESZ.

(Communicated at the meeting of February 24, 1940.)

§ 1. *Introduction.*

For the further investigation of the results of the gravity expeditions with submarines of the Netherlands Navy, the writer has prepared new tables for regional isostatic reduction; he has applied the system published in the Bulletin Géodésique N°. 29, 1931, to five different degrees of regionality. This system distributes the isostatic compensation of each topographic element over a wider area than the area of the element itself and the distribution in a horizontal sense follows the rule that the amount of compensation is proportional to the vertical displacement brought about by the bending of the Earth's crust under the weight of the topographic elements. For each element it adopts the bending-curve of the solution of HERTZ for the bending of an elastic plate floating on a fluid and submitted to a concentrated surface load. Introducing the length l given by

$$l = \sqrt[4]{\frac{m^2 E h^3}{12 (m^2 - 1)(\theta_1 - \theta_0) g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

E = modulus of elasticity of the Earth's crust,
m = coefficient of Poisson,
h = thickness of the Earth's crust,
$\theta_1 - \theta_0$ = difference of the specific densities of the plastic substratum and the topography,

the central part of the bending-curve of HERTZ was used up to a distance of 2.905 l from the concentrated load; the outward waves of small amplitude at greater distances from the center were neglected. The adopted curve is represented by fig. 1. So for each topographic element the isostatic compensation is assumed to be distributed in a horizontal sense up to a distance of $t = 2.905\, l$, having a maximum specific density in the center and this density falling to zero towards the border t according to the curve of fig. 1. The tables have been made for five different values of l, i.e. 10, 20, 40, 60 and 80 km corresponding to values of the outer boundary t of 29.05 km, 58.10 km, 116.20 km, 174.30 km and 232.40 km. They have been published in the Bulletin Géodésique N°. 63, 1940 and details about

these laborious computations may be found in "Verhandelingen der Kon. Akad. v. Wetenschappen, Afd. Natuurkunde", 1ste Sectie, Dl. XVII, N^0. 3. The Netherlands Geodetic Commission has defrayed the expenses of the computations.

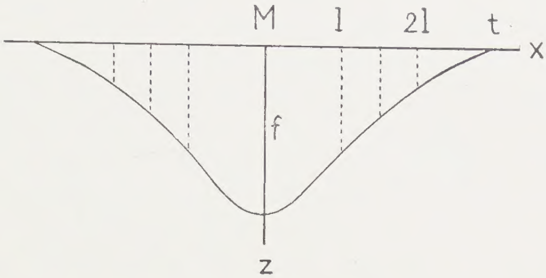


Fig. 1. Curve of the distribution of the regional isostatic compensation.

In these fundamental tables the compensation has been assumed to be distributed uniformly over a depth H below sea-level and the tables comprise all values of H from 0 to 60 km. From these tables other tables have been derived for practical use in which the compensation extends from a depth of 30 km down to a depth of $30 + 4.45\,s$, s being the height of the topography above sea-level. This last assumption corresponds to that of the Airy-Heiskanen tables for local isostatic reduction based on the principle of a crust of a normal thickness of 30 km and a density of 2.67 floating on a plastic substratum of a density of 3.27, each element being in hydrostatic equilibrium independent of its surrounding and sinking till a local root at the lower boundary of the crust is formed reaching from a depth of 30 km down to $30 + 4.45\,s$ km. The new tables, therefore, adopt the same system of hydrostatic equilibrium of the crust but they assume the root at the lower boundary of the crust to be broader according to a bending of the crust, instead of its sinking locally under the load of the topography. Incase we have a submarine topography, we have to introduce a value of s with a negative sign, the topography being now below sea-level, and we have to put $s = -0.615\,d$, if d is the depth, in order to take into account that the deficiency of mass has a specific density of 2.67—1.03; 1.03 has been assumed for the specific density of sea-water. The tables will be published in the course of this year.

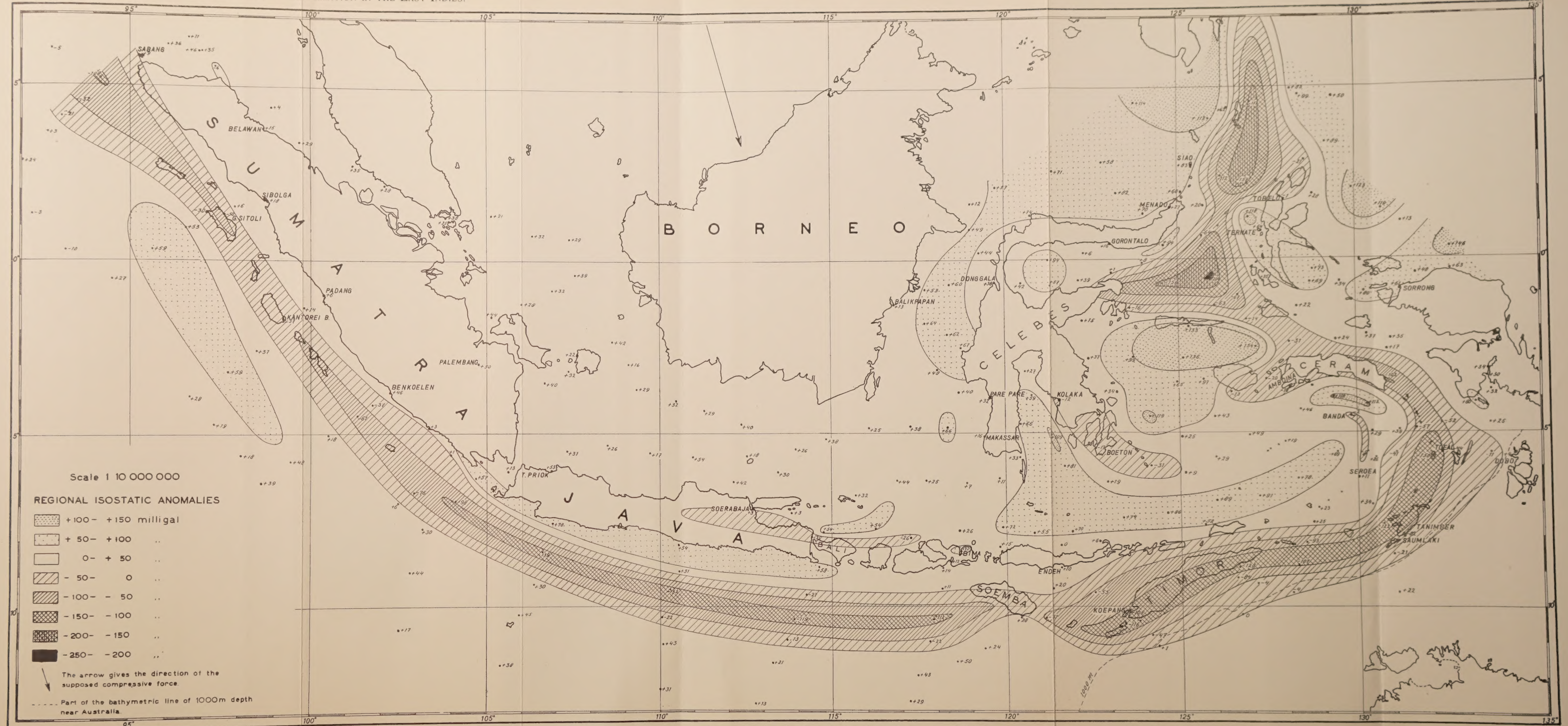
The Netherlands Geodetic Commission has used these tables for the isostatic reduction of all the gravity values obtained at sea. This paper contains a provisional publication of some conclusions derived from the results of these reductions for the East Indies. For each station six anomalies have been deduced, five according to the five degrees of regionality that have been mentioned and one for local compensation under the same conditions of vertical distribution. These last values as well as those for small regionality give an irregular picture of the distribution of the anomalies over the Archipelago; the anomalies for the three largest

degrees of regionality show simpler features. So probably a large degree of regionality fits better the actual conditions in the Earth's outer layers. The most regular picture seems to be given by the anomalies computed with the tables for $l = 60$ km, which corresponds to an outer radius t of the compensation of 174.3 km. They are represented by the map of this paper which shows the isanomales for intervals of 50 milligals and as an example two gravity profiles are given in fig. 2*a* and *b*. The maps and profiles for $l = 40$ km and for $l = 80$ km also show acceptable distributions. A later publication by the Netherlands Geodetic Commission will contain the maps of the anomalies and a great many profiles for all the values of l and will thus allow a comparizon. The conclusions mentioned in this paper will be derived from the present map but nearly all of them would remain valid when the maps for $l = 40$ km or $l = 80$ km had been used.

§ 2. *The Map of the Regional Isostatic Anomalies ($l = 60$ km, $t = 174.3$ km).*

In considering this map the writer thinks that there are several reasons why an attempt is justified to explain the anomalies by taking as a base the simple mechanical picture of a rigid Earth's crust floating on a plastic substratum and subject to deforming forces; the present publication in fact is such an attempt. Of course he is perfectly aware that the real conditions in the Earth are more complicated. In the first place he will assume only one layer of constant density floating on a denser substratum of a specific density of 0.6 more than that of the crust. This is certainly not true as seismic and other evidence points to at least two and probably more rigid layers of increasing density downwards, but this simplification does not make a difference in principle for the main conclusions; they can be adapted without difficulty to the more complicated picture of more than one rigid layer. In the second place changes of state, e.g. from the crystalline to the amorphic, or differentiations and chemical reactions may occur. There is no doubt that these phenomena may affect the main occurrence; a further research may decide how far this will be the case. The object of this study is to see how far the above simplified picture will go in explaining the facts.

As a first reason justifying this attempt the writer may mention the fact that the regional isostatic reduction, based on this assumption, undoubtedly simplifies the anomaly map and so the way the topography is compensated is obviously in harmony with it. He must refer to the future publication of all the maps and profiles for proving this statement. For a second reason the writer may bring forward the success of the buckling hypothesis, which he advanced in 1930 for explaining the main feature of the anomaly map, the narrow belts of strong negative anomalies through the Archipelago. Although other attempts for an explanation have been made, it still remains the one that has found the widest support. It is based on the same mechanical picture and we shall now have to consider it a



Regional Isostatic Gravity Anomalies in the East Indies ($l = 60$ km, $t = 174.30$ km)

little more in detail as it is in fact one of the principal parts of our attempt at an explanation of the anomaly field.

As we shall point out in the next § an elastical plate floating on a fluid of greater density and submitted to a horizontal compressive force, will remain straight as long as this force is below a certain limit D_0 which we may call the buckling limit. When this limit is passed, waves will form and a slight increase of the force will then suffice for these waves to increase further in amplitude till one of them buckles. The writer assumes this to have occurred in the geosyncline in the East Indies under the effect of the tectonic forces. This corresponds to the great folding and overthrusting that has been found in the islands in the belt of anomalies. The buckling takes place in a downward sense and thus a great root of light crustal material is formed at the lower boundary of the Earth's crust. Because of its displacing the denser substratum this represents a large defect of mass and so the strong negative anomalies may thus be explained. We may adopt the supposition that a downward wave buckles downwards and not an upward wave upwards, because of the fact that the energy needed for the first is proportional to the difference of density of the substratum and the crust, which may be estimated at 0.6, while for the latter it must be proportional to the density of the crust of 2.7. This theoretical consideration is confirmed by the beautiful experiments of Dr. PH. H. KUENEN [1]), who compressed a plastic layer floating on water; gradually the layer forms waves and at a certain moment one of these waves buckles downwards. These experiments give a good support to our hypothesis. We may assume that in a further phase of the orogenetic development the upper layer of the crust will crumple upwards and gradually form a ridge that in its ultimate shape will compensate the defect of mass formed by the root. Thus the stage is reached of the complete mountain range with its compensation by a root of crustal material, about $4\frac{1}{2}$ times as large as the range itself, which we find in the Alps and other folded mountain ranges. The main point of this hypothesis, therefore, is that the root is formed before the range itself and that this stage is now reached in the East Indies. The formation of a surface ridge is already going on in the East Indies; in most parts the belt of anomalies coincides with a submarine ridge or with islands as is shown e.g. by the profiles of fig. 2*a* and 2*b* but this ridge is still far too small to compensate the negative anomalies caused by the root. The formation of this surface ridge may also be brought about by the rising of the buckling zone because of the readjustment of the isostatic equilibrium or by other causes; we shall come back to this point. The buckling hypothesis may also explain another feature, i.e. the well-known fact that an orogenetic cycle begins by the formation of a geosyncline; this may be interpreted as the downward

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[1]) PH. H. KUENEN, The negative isostatic anomalies in the East Indies. Leids. Geol. Meded. VIII, 2 (1936).

wave which afterwards, when the compression continues, buckles inwards.

These few words may suffice for giving an idea of the main points of

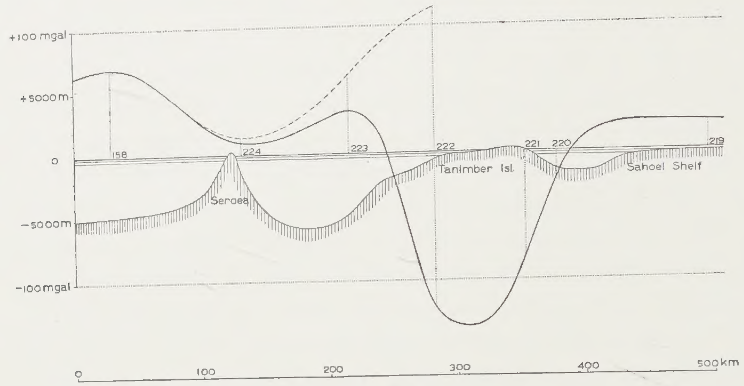

Fig. 2*a*. Gravity profile over Seroea and Tanimber Isl, Drawn line: regional isostatic anomalies ($l = 60$ km, $t = 174.30$ km), Dotted line: the same anomalies after subtracting the estimated effect of the root, Gravity Stations indicated by their number.

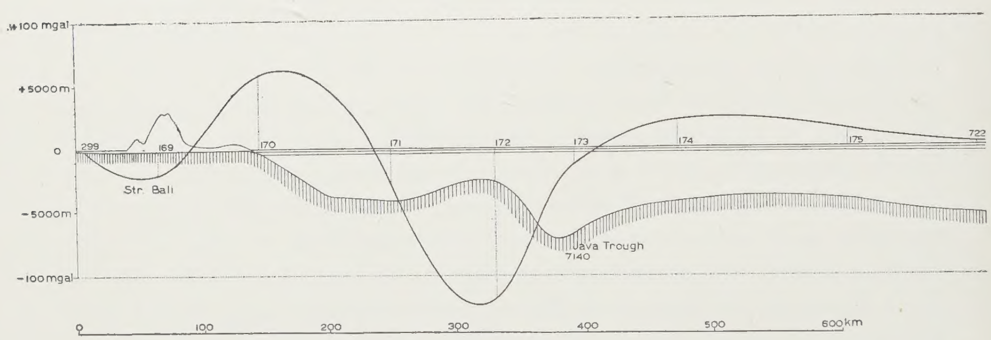

Fig. 2*b*. Gravity profile over Str. Bali southwards, Drawn line: regional isostatic anomalies ($l = 60$ km, $t = 174.30$ km), Gravity Stations indicated by their number.

the buckling hypothesis. The writer should now wish to look at the distribution of the phenomenon through the Archipelago. In the first place we may notice on the map that the anomalies point to a compressive force mainly working in one direction as indicated by the arrow, and not to stresses working in all directions. Where the anomaly belt cuts this direction under an angle of at least 45° the anomalies are large, but where the angle is small the anomalies are likewise small, as e.g. in the profiles west of Sumatra, in the profile from the Banda Sea towards New Guinea and also in the E.W. profile over Strait Surigao, north of Mindanao; this last profile outside the area of the map does not show negative anomalies at all. As the total of the negative anomalies in a profile over the belt is a measure for the size of the root that has formed at that place and, there-

fore, also for the amount of the shortening of the crust, this shortening must have been smaller for these last profiles than for the profiles elsewhere. This finds an explanation by the assumption of the compression working in the indicated direction because in that case the component of the compressive force in the sense of these profiles is small. The writer considers it a favourable point for the mechanical picture advocated in this paper that these differences in intensity of the negative anomalies along the belt may thus be explained by a simple supposition.

The foregoing considerations about the anomaly-belt have already been expressed by the writer in previous papers; in what follows he wishes to add a new view-point partly based on the results of the new isostatic reductions. These results have led viz. to a slight alteration of the isanomales near the island of Soemba. The anomalies west of the island of Timor bring about a pointed course for the isanomale of —50 milligal near the S.E. Cape of Soemba and this has brought the writer to draw the isanomales in that neighbourhood as they are given on the map. This leads to a new vision on the irregularity of the belt of anomalies in this area. While the old anomaly map showed an interruption of the belt, the new one only points to a shift of the axis. This is better acceptable from the view-point of the buckling-hypothesis; an interruption would be difficult to understand because the question rises how the crust could be shortened to both sides of the interruption without the block of Soemba taking part in this shortening. A shift of the axis along a line which is about in the direction of the compression could, however, be better explained. In this connection it is worth while to notice that this shift occurs exactly where the Australian continent begins to face the belt; for showing this the map has been provided with the contour line of 1000 m depth. This coincidence suggests a causal relation and it is easy to think of a satisfactory explanation. Considering the great amount of sedimentation at the continental edge we could understand that a continent can be surrounded by a belt of weaker crustal material and in that case the buckling axis could not continue from its course south of Java straight towards the east but it should have to shift towards that zone of weakness; in its further course to the east it continues to follow the continental edge. We thus get a satisfactory explanation of the curious twist of the island ridge and of the anomaly belt in this neighbourhood. Here again the mechanical view-point provides us with a simple hypothesis. The fact that the island of Soemba does not show great folding or overthrusting is in good corroboration.

For further considerations about the buckling zone the writer wishes to refer to a future and larger publication, which will allow to go more into detail. Here other evidence regarding the mechanical picture advocated in this paper asks our attention. When we look at the anomaly map a periodicity in the occurrence of belts of negative and of positive anomalies strikes us. We notice it when drawing a line from the Celebes Sea towards the S.S.E.; we see here a regular succession of positive and negative

anomalies, more or less in the shape of belts, and the distance of successive features is of the order of 150 to 200 km. In a N.S. profile through Java we find the same: a negative belt in the northern half of Java, coinciding with the oil-syncline, a positive belt over the southern half of the island extending over a strip of the Indian Ocean, the belt of large negative anomalies south of it and lastly a belt of smaller positive anomalies which does not show in the isanomales but which may be derived from the figures. West of Sumatra we find a similar periodicity. And in the West Indies which show a far-going analogy to the East Indies and where likewise a belt of strong negative anomalies has been found, a similar periodicity may be noticed. So it looks as if it is a regularly occurring feature and the question arises what this periodicity may mean.

Here again the writer thinks that the mechanical view-point may provide us with a better explanation than other lines of interpretation would be able to give; the obvious view-point is to interpret these alternating belts as brought about by a wave-formation in the rigid crust. The waves must give alternating deviations from the isostatic equilibrium of the crust, the upward wave causing positive anomalies and the downward one negative anomalies. We may find an important corroboration of this idea in the fact that the wave-length of 300—400 km is in keeping with that derived from the results of the new regional isostatic reduction; in the next § the writer will give a theoretical treatment of the phenomenon which will show the agreement. Another corroboration worth while mentioning is the fact that the profiles of the ocean bottom in many of the gravity profiles west of Sumatra and south of Java show a regular wavelike topography; the amplitude of the wave, i.e. the height of the topography, corresponds to the amplitude of the wave in the gravity anomalies when assuming no isostatic compensation and this is of course as it ought to be according to our supposition because the wave is entirely a deviation from the isostatic equilibrium. We do not find this same relation to the topography as clearly elsewhere in the Archipelago, but this may be explained by the complicated deformations of the surface layers and by the other surface effects of erosion, sedimentation and volcanological activity. The writer must again refer to the future publication which will contain all the gravity profiles for the corroboration of these statements; an instance may be seen in fig. 2*b*, a gravity profile going S.S.W. from Strait Bali. He may also refer to that publication for a detailed comparizon of this interpretation with the old attempt made in 1934 in "Gravity Expeditions at Sea", Vol. II [1]), where he attributed the positive anomalies in the deep basins, e.g. the Banda Sea and the Celebes Sea, to more or less local convection currents in the substratum and the belts of positive anomalies south of Java and west of Sumatra to an upwards drag of the crust because of the regional readjustment of the isostatic equilibrium of the crust over the belt of strong

1) Publ. Neth. Geod. comm., Delft 1934.

negative anomalies. Both explanations meet with difficulties when we go deeper into the matter and so the writer considers the present attempt not only as simpler but also as better.

The assumption of waves in the crust in the present period implies the assumption of a still working compressive force. This is in harmony with the fact that in a very recent period, in the pleistocene, folding has occurred in the oil-geosynclines of the Archipelago and also in E. Celebes and Boeton. It is also in keeping with the strong seismicity of the Archipelago. So we assume the crust to be subject to a compressive force in a horizontal sense, but we know that it is also subject to an upward force in the belt of strong negative anomalies because of the tendency to restore the isostatic equilibrium [1]). We have, therefore, to take up the problem of the deformations of the crust by this combination of forces. We shall take this up in the next § and we shall have a look at the consequences of these results for the recent geology in the ensuing §.

§ 3. *Formulas for the deformation of a floating elastical plate under the effect of a vertical force and of horizontal compression.*

We assume the plate to have constant thickness and infinite dimensions. We further assume the problem to be two-dimensional and we shall here only treat of the simplest case of only one concentrated vertical force in upward direction. We put the origin of the coordinates in the point where the force acts and we take the horizontal axis as the X axis and the vertical one as the Y axis (fig. 3). We assume the deformations of the plate to be small enough that we may neglect the second and higher powers of dy/dx. In these assumptions we have neglected the curvature of the Earth, but it is easy to see that its only effect is an upward shift of the plate of $D/\theta_1 g R$, where D is the horizontal compressive force, θ_1 the specific density of the substratum and R the Earth's radius. Using the same letters and indications as introduced on the first page of this paper and introducing the quantities l_1 and D_0 given by

$$l_1 = \sqrt[4]{\frac{m^2 E h^3}{12(m^2-1)\theta_1 g}} \quad \ldots \ldots \ldots \quad (2)$$

$$D_0 = 2\,\theta_1 g\, l_1^2 \quad \ldots \ldots \ldots \ldots \quad (3)$$

we find the following differential equation for the displacement y of the plate in vertical sense

$$l_1^4 \frac{d^4 y}{dx^4} + 2\,\frac{D}{D_0}\, l_1^2 \frac{d^2 y}{dx^2} + y = 0 \quad \ldots \ldots \ldots \quad (4)$$

We see at once that for $D = D_0$ the solution shows a simple sine-curve and a further investigation by means of no longer neglecting the second

[1]) It is likewise subject to the vertical forces exerted on the crust by the topographic features but these forces are smaller and we shall neglect them here.

and higher powers of dy/dx shows that the amplitudes are quickly increasing when D increases slightly. So we see that D_0 is the buckling-limit of the Earth's crust, i.e. when D reaches this limit the deformations are growing so large that one of the waves will give way and a root will form. In the further investigations we shall suppose $D \leqq D_0$ and for simplification we shall introduce the angle β given by

$$\frac{D}{D_0} = \cos 2\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

So β may vary from 45° for $D=0$ to 0° for $D=D_0$.

The quantity l_1 has the dimensions of a length in the same way as the quantity l introduced on the first page by formula 1. When, however, the crustal wave is formed below the sea-surface we have not to introduce θ_1 in formula 2 but $\theta_1 - 1.03$, i.e. diminished by the specific density of sea-water. If we indicate the length we thus get by l_2, we have the following relation between the three values

$$l_1 = \sqrt[4]{\frac{\theta_1 - \theta_0}{\theta_1}}\, l \qquad l_2 = \sqrt[4]{\frac{\theta_1 - \theta_0}{\theta_1 - 1.03}}\, l \quad . \quad . \quad . \quad (6)$$

and introducing $\theta_1 - \theta_0 = 0.6$ and $\theta_1 = 3.27$ we find

$$l_1 = 0.656\, l \qquad l_2 = 0.719\, l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6A)$$

So the value of $l = 60$ km which has been adopted for the anomalies of the map, corresponds to $l_1 = 39.4$ km and $l_2 = 43.1$ km.

After introducing formula 5 in the equation 4, we find the general solution of 4 in the following shape

$$y = A e^{-ax} \cos(bx + \varphi) + A' e^{ax} \cos(bx + \varphi') \quad . \quad . \quad . \quad (7A)$$

in which A, A', φ and φ' are integration constants and

$$a = \frac{\sin\beta}{l_1} \qquad b = \frac{\cos\beta}{l_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7B)$$

When D does not reach the buckling-limit, i.e. when $a > 0$, this formula represents two waves of the same length of which one is increasing with x and the other diminishing. These waves are caused by other forces working on the crust besides the compression D and obviously the constants A and A' must be chosen in such a way that the waves are dying out to both sides of the outward force which causes them. If there is no outer force working on the crust, A as well as A' are zero and the crust does not undergo any deformation save of course the thickening by elastical compression; it remains plane. In our case of one vertical force working in the origin of the coordinates, we have the following set of formulas for y and its derivatives for positive values of x, representing a wave dying out

for greater values of x, and for negative x we find the same curve towards the side of the negative x.

$$y = A e^{-ax} \cos(bx + \varphi) \quad \ldots \ldots \ldots \quad (8A)$$

$$\frac{dy}{dx} = -\frac{A}{l_1} e^{-ax} \sin(bx + \varphi + \beta) \quad \ldots \ldots \quad (8B)$$

$$\frac{d^2 y}{dx^2} = -\frac{A}{l_1^2} e^{-ax} \cos(bx + \varphi + 2\beta) \quad \ldots \ldots \quad (8C)$$

$$\frac{d^3 y}{dx^3} = \frac{A}{l_1^3} e^{-ax} \sin(bx + \varphi + 3\beta) \quad \ldots \ldots \quad (8D)$$

The length L of the waves is

$$L = \frac{2\pi}{b} = 2\pi l_1 \sec\beta \quad \ldots \ldots \ldots \quad (9)$$

and as β varies from 45° for $D = 0$ to 0° for $D = D_0$ the wave-length varies from $2\pi\sqrt{2}l_1 = 8.90\ l_1$ to $2\pi l_1 = 6.28\ l_1$. Taking e.g. the case we shall treat of presently that $D = 0.6\ D_0$ and assuming a submarine wave of the crust and so, according to $l = 60$ km, $l_2 = 43.1$ km, we find $L = 303$ km. This is in good keeping with the wavelength of the anomaly-map as has been stated in the previous §.

For the determining of the integration-constant A we may use the value of the vertical force P_0 working on the crust in the origin. If H_0 is the total shearing-force in a cross-section perpendicular to the crust for x infinitely small but positive, we have, if the index $_0$ indicates values for $x = 0$

$$P_0 = 2 H_0 + 2 D \left(\frac{dy}{dx}\right)_0 = 2\,\theta_1 g\, l_1^4 \left(\frac{d^3 y}{dx^3}\right)_0 + 4\,\theta_1 g\, l_1^2 \cos 2\beta \left(\frac{dy}{dx}\right)_0$$

which gives

$$P_0 = -2\,\theta_1 g\, l_1 A \sin(\varphi - \beta)$$

or

$$A = -\frac{P_0}{2\,\theta_1 g\, l_1 \sin(\varphi - \beta)} \quad \ldots \ldots \quad (10)$$

We may derive the second integration-constant φ from the value of dy/dx for $x = 0$. When the crust had not been broken in that point, we should have had to introduce $dy/dx = 0$. When, however, we identify this point with the buckling zone, other values are also possible, and so we have in

general to introduce a certain given value for $(dy/dx)_0$. We may thus derive φ from

$$\left(\frac{dy}{dx}\right)_0 = \frac{P_0 \sin(\varphi+\beta)}{2\,\theta_1 g\, l_1^2 \sin(\varphi-\beta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

Introducing A and φ in $8A$ we find the curve for y.

In fig. 3 three cases are given, all for the same value of P_0, i.e.

fig. 3a $\quad \left(\frac{dy}{dx}\right)_0 = 0$

fig. 3b $\quad \left(\frac{dy}{dx}\right)_0 = \frac{P_0}{2\,\theta_1\, g\, l_1^2}$

fig. 3c $\quad \left(\frac{dy}{dx}\right)_0 = \frac{P_0}{\theta_1\, g\, l_1^2}$

and for each case three assumptions have been made for D. The drawn curve corresponds to no compression, i.e. $D=0$, the point-dot curve to

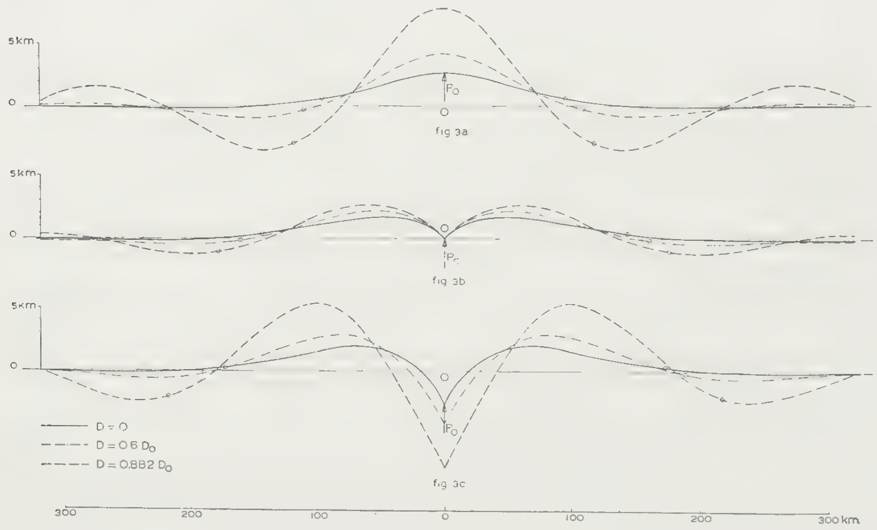

Fig. 3. Theoretical curves of crustal deformation under the effect of an upward force P_0 and a horizontal compressive force D.

$D=0.6\ D_0$ and the dotted curve to $D=0.882\,D_0$. On purpose only positive values of $(dy/dx)_0$ have been chosen, because as a consequence of the downward buckling in a previous period, positive values seem more likely than negative values. They appear also to follow from some of the profiles, e.g. those south of Java of which fig. 2*b* is an instance. The vertical scale of the curves has been chosen in such a way that for the following acceptable values of P_0, θ_1 and l_1 the vertical scale is ten times exaggerated with regard to the horizontal scale, i.e. both scales are the same as those of the topography in fig. 2*a* and 2*b*. For P_0 a value has been

adopted of 7.2×10^{12} gram per cm of the third dimension; this corresponds to a root of a cross-section of 1200 km^2 and a deficiency of specific density of 0.6 and this again is in keeping with the negative anomalies south of Java. For θ_1 and l_1 the values of 2.24 and 43.1 km have been adopted corresponding to a wave-formation below sea-level. Introducing this in formula 3 we find for the buckling-force D_0 8.3×10^{13} gram per cm of the third dimension, i.e. about $11\frac{1}{2}$ times as much as P_0. It is of course possible to introduce different values of $(dy/dx)_0$ to both sides and also to derive equations for more than one vertical force working on the crust, but the writer must refer to a future publication for these more complicated solutions.

We may derive interesting conclusions from fig. 3. In the first place we see that an increase of the compressive force D may bring about rising as well as sinking for the central zone, depending on the slope of the crust to both sides. Fig. *3b* gives a case where this zone does not move at all. In all cases the waves to both sides increase in amplitude when D increases. When $D = 0$, i.e. when there is no lateral compression, these waves are insignificant and they become larger and larger when D increases. When D is near the buckling-limit they become nearly as large as the one in the central zone.

So we see that the effect of a change in the compression, even incase the force P_0 is kept constant, shows a great diversity, depending on circumstances, and this diversity is still more striking when we include cases where the slope of the crust to both sides of the central zone differs. This rather surprising diversity provides us with a means to adapt this solution to many different cases of behaviour of the crust and to bring them all back to one common origin. To point out these possibilities is one of the aims of the writer in this paper. In the next § he will make use of these possibilities for a few cases in the East Indies but an exhaustive treatment is out of the question here; the writer must defer this to a future occasion. He wishes to lay stress on the fact that the basic assumption opening these possibilities is a simple and acceptable one, the supposition of a compressive force working in the crust; the presence of the vertical force P_0 is no assumption but an observed fact directly derived from the gravity anomalies.

With a view to an hypothesis he wishes to make in the next § the writer has marked the points of maximum curvature in the downward waves of the curves of fig. 3. It is easy to derive their location from the equations 8 by equalling $8D$ to zero which gives

$$x = \frac{n\pi - \varphi - 3\beta}{b} \quad \ldots\ldots\ldots \quad (12)$$

For $D = 0$ these points of maximum curvature are farthest away from the deepest points of these downward waves; for increasing D they get

nearer to these points and they coincide with them for the pure sine-curve which is obtained for D equal to the buckling limit.

§ 4. *Geological interpretation of the equations of § 3.*

According to the summary given by J. H. F. UMBGROVE in "Geological History of the East Indies" [1]) the recent geological history of the islands of Timor, Tanimber, Kei and Ceram, all situated in the belt of strong negative anomalies, is the following. After the last period of strong folding and overthrusting in the upper miocene, a period of denudation of the elevated islands and subsidence followed so that in the pliocene the sea could invade them. Then trough faults and "graben" have been formed in the axis of the islands and about parallel to them, and finally towards the end of the pleistocene elevation occurred again and faulting. At the same time probably of this last elevation the troughs to both sides of these islands have sunk away.

In the light of the results of the previous § we may perhaps try to explain these occurrences by changes in the compressive force D. A decrease of this force may have brought about the subsidence in the first part of the pliocene and this is in keeping with the fact that in no part of the Archipelago folding occurred in that period. The writer does not think that the compressive force can have disappeared entirely because we may surmise that in that case the belt of negative anomalies would have succeeded in restoring its isostatic equilibrium. As since that time there has not occurred enough folding in this zone to give an easy explanation for these great anomalies, we must probably keep to the idea that they have at least for the greatest part originated during the great folding period in the upper miocene or earlier, and this probably implies that in no period since then the compression has disappeared entirely.

Towards the end of the pliocene or the beginning of the pleistocene folding begins to occur again in the Archipelago, e.g. in the oil-geosynclines, in Boeton and in E. Celebes and this appears to point to an increase of the compressive force. There seems reason to bring this in connection with the formation of the "graben" on Timor and the other islands and to attribute this likewise to an increase of the compressive force; such a reaction of the old buckling zone does not seem difficult to explain. The great vertical movements towards the end of the pleistocene indicate a further increase of the compression, which according to the results of the previous §, must gradually have brought about the waves that in the present period cause the alternating belts of positive and negative anomalies that have been found. The rising of the islands of Timor, Ceram etc. and the sinking away of the troughs to both sides could thus be attributed to this wave-formation; the deformation there would,

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[1]) Bulletin of the American Association of Petroleum Geologists, Vol. 22, 1 Jan. 1938.

therefore, have to belong to the type of fig. 3*a* or between fig. 3*a* and 3*b*

This explanation of the vertical movements makes it possible to understand that in other parts of the central zone slighter or even no rising has occurred. This may explain the great differences of elevation in this zone, the alternation of islands and lower parts between. South of Java we have a particularly low part of the zone and so we should have to classify this part under the types of fig. 3*b* or perhaps even 3*c*. We may find two remarkable confirmations of this view-point. In the first place the topographical profiles, as e.g. that of fig. 2*b*, show a downward slope towards the central part and so a value of $(dy/dx)_0$ leading to one of these types may be expected. As the horizontal and vertical scales of fig. 2*b* and fig. 3 are the same, we may directly compare the slopes in both figures. In the second place we find zones of positive anomalies at rather large distances of the central zone to both sides, of which that over the southern part of Java is one, and this is in better agreement with fig. 3*c* than with fig. 3*a*.

On the other hand the profile of fig. 2*a* over the Tanimber Islands comes near to the type of fig. 3*a* and this is in keeping with this part of the central zone having risen to a higher position than that south of Java. For making this clear the writer has subtracted from the regional anomalies of part of the profile, as given by the drawn curve in fig. 2*a*, the negative anomalies caused by the root of crustal material and the resulting anomalies are given by the dotted curve of fig. 2*a*. These anomalies may thus be assumed to be caused by the wave-formation of the crust and the curve in fact does fairly well agree with the type of the curves of fig. 3*a*. The writer considers this dotted curve as a first and still rather uncertain trial to free the anomaly-curve from the effect of the root. As we do not know its exact dimensions and shape this effect may be different and so the remaining part of the anomalies would also be different; this would mainly affect the part over the central zone. In consequence of this it might be possible that a nearer consideration would put the type of the dotted curve between that of fig. 3*a* and fig. 3*b*. It could not, however, quite be of the type of fig. 3*b* or still further away from fig. 3*a*.

In any case the inner Banda arc, represented in this profile by the island of Seroea, would coincide with a downward wave and thus it would correspond to the downward wave that in the profiles over Java and Sumatra, e.g. in the profile of fig. 2*b*, represents the oil-geosyncline. In view of the fact that both features occur in the same topographic axis, the axis of the greater and smaller Sunda Islands, this seems acceptable. The question arises whether we could thus understand the presence of strong volcanic activity in this axis. The writer is inclined to answer this question in the affirmative; a downward wave of the crust must bring about tension in the lower half of the crust or at least a diminishing of the pressure and this might explain the rising of magma. If this explanation is correct the location of the volcanoes ought more or less to coincide with the location

of the maximum curvature of the crustal wave and this has led the writer to indicate these spots in the curves of fig. 3. We see that they are not found in the axis of the downward wave but shifted towards the side of the central zone and more so for smaller values of the compressive force, i.e. for waves that die out quicker with the distance to the central zone. For the profiles over Java this seems to be more the case than for the profile over the Tanimber Islands and the location of the volcanoes to the south of the oil-geosyncline appears in good keeping with this surmise. Besides this view-point the writer still thinks that the supposition he has formerly [1]) made regarding the occurring of volcanoes in belts on the inside of curved parts of the orogenetic belt, has a bearing on the subject, especially to explain that no volcanoes occur on the outside of a curve. This problem of the distribution of the volcanic activity in the Archipelago is, however, a complicated one, requiring much expert knowledge, and so the writer does not think he can do justice to it, certainly not in the few lines he can devote to it here.

An important point has still shortly to be mentioned. In his paper on "The negative isostatic anomalies in the East Indies" KUENEN mentions a problem brought forward by MAC GILLAVRY as an objection to the buckling hypothesis, i.e. the question how it is possible that the mean elevation of the eastern half of the Archipelago is below sea-level and probably lower than it has been in former geological periods, although the sial crust has been compressed; it looks as if this ought on the contrary to have brought about a rising of the mean elevation. There is no doubt that if the profile of fig. 2*a* over the Tanimber Islands should have to be identified with the deformation of the crust as given by fig. 3*a*, we must assume that the undeformed position of the surface should have to lie at a depth of some 5000 m or more. And so it appears as if there has been a considerable sinking of this whole area in recent periods. It is a great problem to find a cause for such a phenomenon which the writer is not able to discuss here, but he may shortly mention the possibility that this sinking could be explained by assuming a great convection-current system in the substratum with its descending current under this part of the Archipelago and rising currents under Asia and Australia or under one of these continents. Because of the drag exerted on the crust by this current such a current could at the same time explain the compressive force in the Archipelago.

The writer should not want to conclude this paper about the mechanical conception for explaining the gravity field and several geological features in the Archipelago, without mentioning also a serious difficulty for adopting it. The question is that the forces and stresses which must be assumed to work in the crust are considerable and larger than is generally supposed to be possible. If we adopt a value for D of 0.6 D_0 and if we

[1]) F. A. VENING MEINESZ, Gravity Expeditions at Sea, Vol. II, pp. 112 and 125, Publication of the Neth. Geod. Comm. Delft, 1934.

take for D_0 the value derived in the previous §, we find the value of D to be 5×10^{13} gram per cm of the third dimension. Supposing the crust to have a thickness of 40 km we thus obtain a mean value for the compressive stress of 12500 kg/cm² and we get still larger figures for those parts of the crust where the bending causes additional compression. The writer cannot go deep here in the question at what figures we have to estimate these additional bending stresses and so he has to reserve that for a future occasion, but he wishes shortly to mention that they would become enormous if we kept to our simplified conception of the crust being one elastical plate. There is no doubt we have to come to a view-point nearer to the actual conditions in the crust for the attack on this question. In two ways we can understand that the bending stresses would be less. In the first place we may assume the crust to be composed of several layers which can more or less slide over each other and in the second place we can presume that the deformation for these large stresses, at least for parts of the crust, has not been of the elastical type but of the plastical or the block-faulting type. For taking up these problems we have to write the equations 1 and 2 in a more general shape by substituting

$$f = \frac{m^2 E h^3}{12(m^2 - 1)}$$

in which f is in general the ratio of the moment of force M working in the crust to the value $\frac{d^2y}{dx^2}$ of the curvature. We thus get

$$M = f\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad (13A) \qquad l = \sqrt[4]{\frac{f}{(\theta_1 - \theta_0)g}} \quad . \quad . \quad (13B)$$

The equations 6 for the values of l_1 and l_2 remain the same and also all the other equations and formulas.

The writer cannot go further in these problems here and so he wishes to conclude his paper with the general remark that this attempt for giving a mechanical explanation seems successful for many questions which are otherwise difficult to explain or to bring under one common view-point, but that the stresses and strains implied are large. He thinks, however, that it will not be easy to escape such a conclusion.

The writer wishes to express his thankful acknowledgements to his colleagues UMBGROVE, KUENEN, ESCHER and RUTTEN for their helpful discussions; many view-points mentioned here may in fact have originated in these discussions, especially in the long talks he had about these matters with the former two. He wants further to acknowledge the assistance of Mr. J. VAN DIJK of the geological Institute of Utrecht for the making of the map and drawings. Lastly he wishes to recall that the whole gravity material is due to the great cooperation of the Netherlands Navy and to the helpful assistance of the Captains MANTE and VAN DER KUN, of the Officers and of the men of the submarines on board of which the observations have been made.

Physics. — *La pression du toit sur le charbon près du front dans les exploitations par tailles chassantes.* Par F. K. TH. VAN ITERSON. (II).

DEUXIÈME CHAPITRE.

Répartition des pressions après écrasement du charbon.

(Communicated at the meeting of January 27, 1940.)

§ 4. *La pression du toit sur les piliers de charbon. Il sera tenu compte des propriétés différentes de charbon et schiste, les deux considérés comme matières élastiques.*

Dans le paragraphe précédent nous avons considéré le rocher, dans le sein duquel des galeries sont creusées, comme un milieu isotrope, homogène et élastique. Nous le supposons chargé en dessous de la limite d'écoulement et acceptons que la loi de HOOKE est applicable.

Mais les propriétés élastiques du charbon et du schiste sont tellement différentes que la répartition des pressions autour d'une galerie creusée dans la veine de charbon sur toute son épaisseur, ne peut pas être traitée comme nous le faisions pour les galeries en terrain homogène. Pas plus les pressions dans le toit et le mur de schiste au-dessus et au-dessous d'un pilier de charbon, ne se répartissent de la même manière qu'autour des piliers laissés en terrains homogènes.

Pour prouver cette thèse, il faut maintenant indiquer en quoi ces matières se distinguent. Plus tard nous parlons de la nature cassante du schiste partout dans la mine où il est mis à nu et de sa plasticité quand il est caché à notre vue et soumis dans tous les sens à une pression élevée mais inégale selon les axes principaux. Nous ferons mention de la fragilité du charbon, qui après rupture se comporte comme une matière incohérente, mais ici nous donnons des résultats d'essais sur la résistance à la pression, le coëfficient d'élasticité et de contraction latérale des deux matières en question.

LÉON MORIN, Quelques effets de pression de terrains. Bulletin de la Soc. de l'industrie Minérale 1912, p. 29.

L'écrasement du charbon se produit		à	40	kg par cm^2
,,	des schistes tendres	à	80	,, ,, ,,
,,	,, ,, moyens	à	170	,, ,, ,,
,,	,, ,, durs	à	220	,, ,, ,,

HEISE–HERBST, Bergbaukunde 1932.

Résistance à la compression:

charbon	35— 290	kg par cm^2
schiste	250— 850	,, ,, ,,
grès	290—1840	,, ,, ,,

MÜLLER, Glückauf 1930:
schiste = 560 ⊥ 815 kg par cm²
grès = 770 ⊥ 800 ,, ,, ,,
PHILIPS, Annales des Mines de Belgique. Les roches houillères 1938, p. 564.
⊥ normal à la couche = parallèle à la couche.

charbon dur anglais		208— 243 kg par cm²
schiste	⊥	400— 800 ,, ,, ,,
,,	=	260— 420 ,, ,, ,,
grès	⊥	700—1400 ,, ,, ,,
,,	=	450—1300 ,, ,, ,,

Ne ferait-on pas mieux de renoncer à l'élaboration d'une théorie apte à réconcilier la mécanique appliquée avec l'art de l'exploitation de mines quand on voit une telle divergence dans les propriétés des matériaux? Les difficultés s'entassent quand on se rend compte que la stratéfication peut être irrégulière, les couches varient d'argile plastique à cuerelles dures. Le plissement des couches a changé le pendage, provoqué des dérangements, des failles, ouvert des clivages dans le charbon, transformé le rocher en diaclases et en sus de toutes ces irrégularités géologiques il y a les ruptures, le foissonnement, les dislocations occasionées par les travaux en tailles dont l'influence se fait souvent sentir jusqu'à cinquante mètres des havées et davantage.

Pour démontrer la complexité du problème nous donnons quelques chiffres suggérant une connexion entre les dimensions des cubes d'essais et la résistance à la compression du charbon.

Résistance à la compression

Auteur	Espèce de charbon	Dimensions des cubes d'essai en cm	Résistance en kg par cm²
Rice-Ensiau Bull. Bur. of Mines 1929, p. 303	Anthracite de Pennsylvanie	6.4—25.4 25.4—30.5 ~ 75 ~ 140	175 (6 fois) 141 (3 fois) 57.4 (1 fois) 21.6 (1 fois)
Lawall-Holland Trans. A. I. M. E. 1932	Virginia	7.6	100—300
MÜLLER en Glückauf 1930	Haute Silésie	4.5 4.5	210 165
Bureau of Mines selon MÜLLER en Glückauf 1930	Charbon dur Charbon tendre		125—400 30—150

Tous ces essais sont exécutés sur des cubes entre surfaces frottantes, et par conséquent sont erronés. FÖPPL [1]) et mieux encore Roš et

[1]) FÖPPL, Technische Mechanik, III, p. 71.

EICHINGER [1]) ont démontré qu'il faut diminuer ces chiffres au moins d'un tiers pour avoir la résistance à l'écrasement de la couche.

Chaque mineur connait au charbon en site deux espèces de plans de clivage, les plus manifestes qui s'entrecroisent régulièrement sont dus à la poussée géologique et traversent la veine sous une inclinaison déterminée et les plans de clivage parallèles au front de taille qui sont dus à l'exploitation de la couche. On reconnait que les premiers sont d'origine tectonique parcequ'on les trouve quelquefois remplis de calcite ou autres minérais, et les derniers se produisent sous nos yeux avec un crépitement bien connu. Il nous parait probable que la divergence entre les résultats d'essai comme la dépendance de la résistance à la compression des dimensions du bloc, sont causées par la fissuration des morceaux de charbon.

Quand on ajoute à ces variations de la substance du terrain, les disparités de la pression dues aux poussées tectoniques et plus encore aux affaissements provoqués par l'abatage, on désespère pouvoir exprimer en formules la distribution de la pression autour de la taille. Mais les régularités avec lesquelles les effets de pression se manifestent dans les exploitations nous incitent à continuer l'étude.

Que sait-on des modules d'élasticité E et de la contraction latérale m? On trouve des chiffres chez LOWENS, Glückauf 1933, p. 34—38, MÜLLER, Glückauf 1930, p. 1601, Lawall-Holland Trans. A.I.M.M.E. 1932 et surtout chez PHILIPS, Annales des Mines de Belgique 1938, p. 564. Les résultats des essais sont variables mais pas tellement comme la résistance à la compression. Nous donnons ici quelques chiffres qu'on peut accepter comme moyennes.

Pour le *grès* et le *schiste* E et m sont des fonctions de la pression. On peut accepter

$E = 600.000$ kg/cm^2 aux faibles pressions

$E = 100.000$ kg/cm^2 aux pressions d'écrasement

m diminuant en même temps de 16 à 3,4.

Aussi à quelque distance des exploitations où le rocher est soumis à une pression dans tous les sens $m \backsim 3,5$.

En contraste avec le rocher pour le *charbon* dans la couche ces modules sont des constantes, on peut accepter

$$E = 50.000 \text{ kg/cm}^2 \quad m = 2.$$

Seulement pour les charges très faibles on trouve pour m un peu plus au laboratoire, mais en chargeant davantage m tombe rapidement à 2 et reste constant.

La constatation que $m = 2$, ce qui veut dire que le charbon ne varie que fort peu en volume quand on le comprime, qu'il se comporte comme le caoutchouc dans ce respect, tandis que le rocher au contraire ne se dilate

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1) ROŠ et EICHINGER, Versuche zur Klärung der Frage der Bruchgefahr. Diskussionsbericht No. 28. Nichtmetallische Stoffe. Zürich 1928, p. 14.

presque pas latéralement quand il est comprimé, mène à des conséquences surprenantes pour l'abatage [1]).

Auteur	Matière	Module d'Elasticité E en kg par cm²	Module de Poisson m
LOWENS, Glückauf 1933, p. 34—38	Grès Schiste sableux Schiste argileux	⊥ 450000 = 400000 ⊥ 275000 = 329000 ⊥ 547000 = 637000	3,4
MÜLLER, Glückauf 1930, p. 1601	Grès Schiste Schiste argileux	⊥ 380000 = 510000 600000 744000	
GREENWALD, Journ. appl. Physics 1937	Grès de Pittsburgh Schiste de idem	~ 200000 ~ 200000	
PHILIPS	Grès	100000 — 600000	16 à 70 kg/cm² 7,64 à 430 „
Ann. Mines de Belgique 1938	Schiste	133000 — 400000	50 à 70 „ 3,4 à 840 „
	Charbon	50000 — 58000	2,23 à 70 „ 2,03 à 430 „
Lawall-Holland Trans. A.I.M.M.E. 1932	Charbon de Virginia	⊥ 26000 — 43000 = 12000 — 50000	⊥ 0,59 = 3,28
MÜLLER, Glückauf 1930 [2])	Charbon de Silésie	24000 — 55000	

⊥ veut dire perpendiculaire à la couche.
= veut dire pression parallèle à la couche.

Référons-nous à la figure 1. Là nous avions supposé le charbon comme indéformable, nous y reviendrons, mais pour le moment nous constatons

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[1]) D. W. PHILIPS and A. HACK, The physical properties of coal measure rocks. Seventeenth Annual Report of the Safety in Mines Research Board 1939, p. 66.

[2]) Pour le module d'élasticité de la roche houllieuse on trouve encore quelques chiffres dans:

O. FLEISCHER, Bergmännische Betrachtungen und Versuche über Gebirgsbewegungen im oberschlesischen Steinkohlenbergbau zur Klärung von Gebirgsdruckfragen. Diss. T. H. Breslau 1933.

O. MÜLLER, Untersuchungen an Karbongesteinen zur Klärung von Gebirgsdruckfragen. Diss. T. H. Breslau 1930. MÜLLER donne ici comme moyen E pour le charbon 25000 kg par cm².

K. STÖCKE, H. HERZMANN und H. UDLUFT, Gebirgsdruck und Plattenstatistik. Elastizitätsverzuche an karbonischen Gesteinen Oberschlesiens, Z. Berg-, Hütt.- u. Sal. Wes. **82**, 307—354 (1934). E pour le schiste en moyen 450.000 kg par cm². E pour le grès en moyen 200.000 kg par cm².

que les bases de notre calcul viennent à nous manquer. Nous avions seulement calculé avec l'élasticité du rocher, qui est minime. Pour la couche de charbon au contraire la détente est grande en vertu du petit module d'élasticité et provoque une grande expansion vers le vide et en même temps la veine se contracte beaucoup en épaisseur. Le déplacement du front fait glisser le charbon par les surfaces du rocher et le détache. Mais que signifie $m = 2$?

Mettez un cube ou un cylindre de charbon dans un moule indéformable qui l'englobe parfaitement et mettez la pression p sur la surface libre, alors cette pression règne dans tous les sens dans le charbon et contre les parois du moule. Otez la pression et les parois sont dégagées. On est tenté à croire qu'il en est de même dans la mine. Quand on met à nu la surface du charbon on dégage toit et mur de la pression sur une certaine étendue.

Comme le disait Lord KELVIN: „On ne connait un phénomène que quand il est possible de l'exprimer en formule". Nous tacherons de traiter le problème selon les lois de l'élasticité. Pour ne pas trop compliquer cette tentative nous négligeons le frottement du charbon le long du toit et du mur. L'observation de tous les jours nous apprend que ces surfaces se présentent comme très lisses en général, le charbon y glisse sur une dizaine de cm, souvent les surfaces ont l'air d'être graissées. Parfois même une couche mince limoneuse lubrifie ces surfaces. Nous avons déjà parlé des plans de clivage parallèles à la direction de la pression dus à l'abatage et pour simplifier la représentation mentale nous considérons la couche de charbon comme étant fendue en d'innombrables colonnes juxtaposées. Toujours nous traitons le problème à deux dimensions y et z.

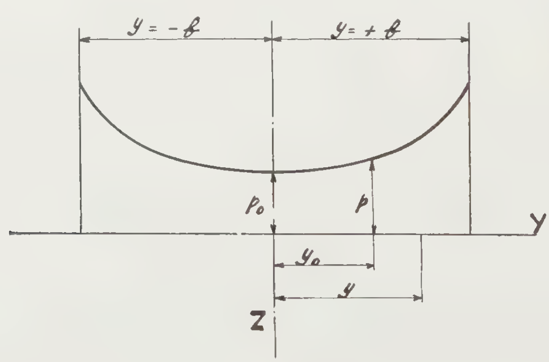

Fig. 11. Notations pour calculer l'augmentation de la pression vers les bords sur un pilier, sur une bande de charbon inexploitée.

Par raisonnement nous croyons que la pression sur le pilier de charbon, laissé entre deux vides (fig. 11), augmente vers les bords. Le cas extrème étant celui de la bande indéformable, ou bien la bande extrèmement mince ou très large, traité dans le chapitre I (p. 99, 1938, figure 10) où les pressions tendent vers l'infini quand on se rapproche des bords.

Il parait impossible de trouver la solution exacte pour la répartition de

la pression sur une bande élastique comprimée entre deux hémi-espaces de matière à autre module d'élasticité; mais on peut chercher une solution serrant entre limites la répartition réelle et alors approcher ces limites.

Nous essayons dans ce but p en fonction de y

$$p = p_0 + k\frac{y^2}{b^2} + l\frac{y^4}{b^4} + m\frac{y^6}{b^6} + \ldots$$

Calculons la déformation de la surface de charbon et celle du mur et nous pourrons choisir p_0, k, l, m, etc. de telle manière que les surfaces s'épousent en autant d'endroits qu'on a d'inconnus, le résultat sera que les deux courbes se confondent. Déjà avec trois inconnus p_0 la pression au milieu et deux coëfficients k et l on obtient un résultat satisfaisant à notre but.

Dans l'annexe 1 nous donnons quelques détails du calcul pour trois cas: $b = a$, $b = 2a$ et $b = 4a$ c.-à-d. largeur du pilier de charbon une, deux et quatre fois sa hauteur et nous discutons à présent les résultats représentés dans les figures 12, 13 et 14.

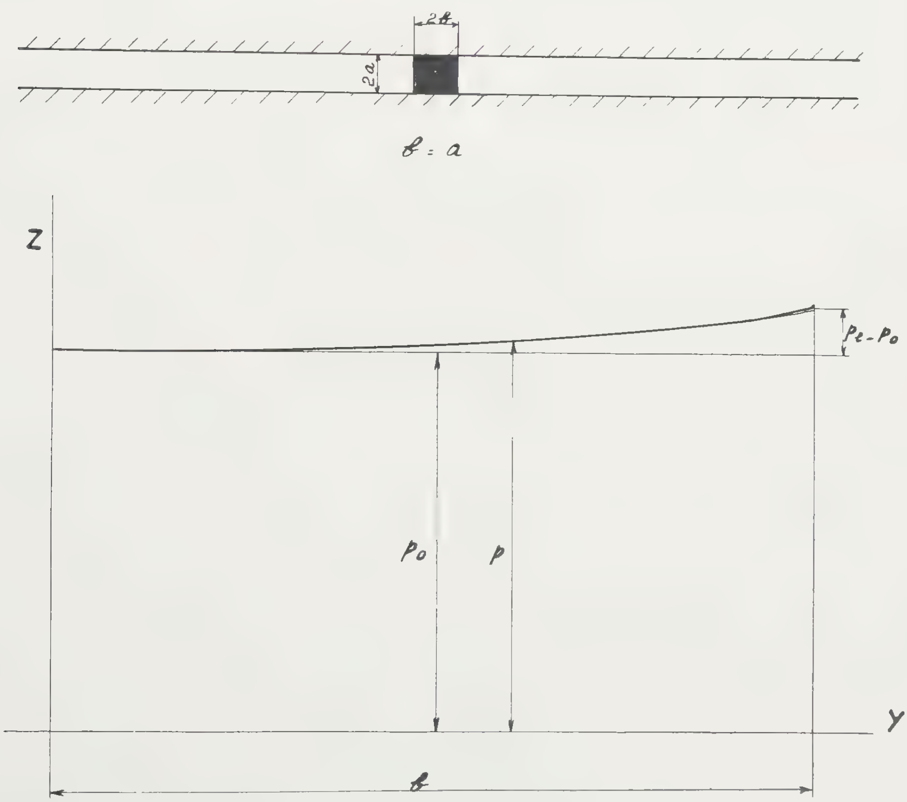

Fig. 12. Répartition de la pression du toit et du mur sur une bande de charbon de largeur $2b = 2a$. E rocher $=$ 360.000 kg par cm^2. E charbon $=$ 60000 kg par cm^2.

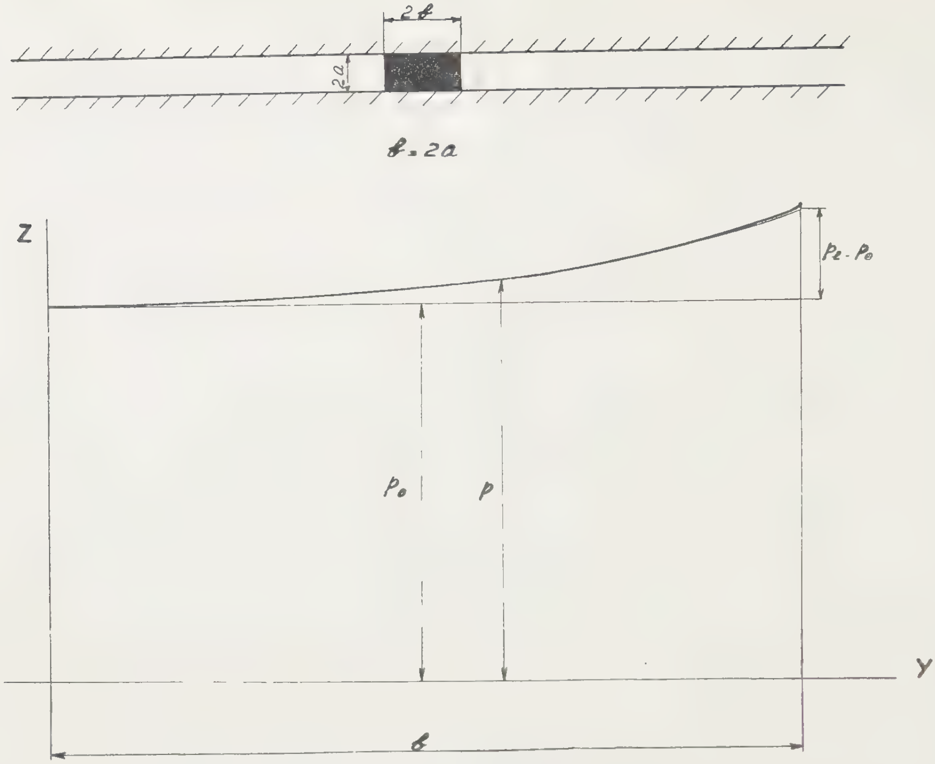

Fig. 13. Répartition pour une bande de charbon de largeur $2b = 4a$.

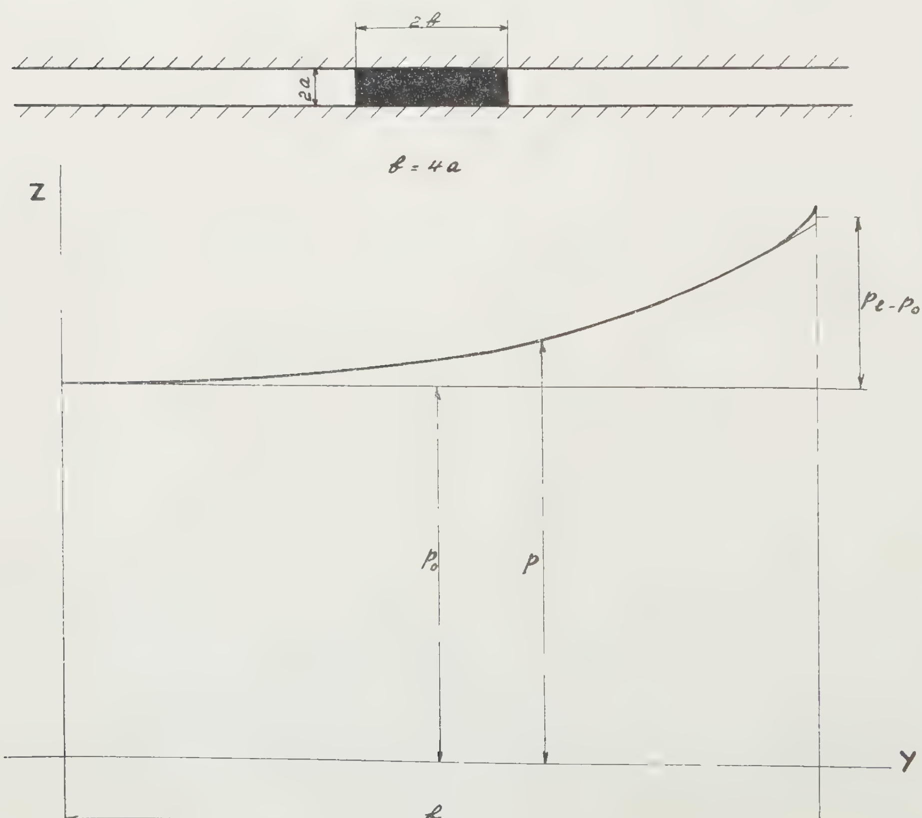

Fig. 14. Répartition pour une bande de Charbon de largeur $2b = 8a$.

Dans chaque figure la courbe inférieure donne la déformation de la surface du charbon à échelle exagérée pour les coëfficients k et l adoptés. Ces déformations, ces raccourcissements sont en même temps une mesure, représentent aussi la distribution de la pression. La courbe supérieure représente l'impression du mur (le sol) sous cette distribution de pression.

On remarque:

1. Au bord même la tangente de la courbe inférieure est peu inclinée et celle de la courbe supérieure prend une inclinaison de 90°. Notre solution ne donne donc pas la répartition mathématiquement exacte des pressions au ras du front de charbon. Mais l'influence locale des coins s'amortit à peu de distance.

2. La répartition exacte de p selon la largeur du pilier de charbon est située entre nos deux courbes, qu'on peut serrer en ajoutant plus de termes à la formule.

3. Pour les piliers relativement étroits que nous avons examinés, avec b une, deux ou quatre fois a, l'augmentation de la pression vers les bords reste limitée, mais elle s'accroît avec la largeur du pilier, ou ce qui revient à la même courbe pour la distribution de pression, l'augmentation vers les bords s'accroît avec amincissement de la couche, jusqu'à devenir infinement grande pour les piliers très larges ou très minces, sauf au ras même du charbon où la solution ne vaut plus sur une étendue très restreinte.

4. Nous résumons nos calculs dans le tableau suivant.

Largeur $2b$ comparée à l'épaisseur $2a$ du pilier de charbon.	Formule approximative pour calculer l'augmentation de la pression vers les bords.	Augmentation de la pression et de la compression du pilier aux bords comparée à celles du milieu. Le rapport réel est entre ces deux chiffres $(p_l - p_0) : p$
$b = a$	$0{,}06\, p_0 \left(\frac{y^2}{b^2} + \frac{y^4}{b^4}\right)$	0,13 et 0,12
$b = 2a$	$0{,}115\, p_0 \left(\frac{y^2}{b^2} + \frac{y^4}{b^4}\right)$	0,252 et 0,23
$b = 4a$	$0{,}22\, p_0 \left(\frac{y^2}{b^2} + \frac{y^4}{b^4}\right)$	0,49 et 0,44
$b = \infty$	∞	∞

5. Le calcul démontre que les pressions augmentent vers le ras du front

de charbon, mais pour les piliers de largeur restreinte dans une mesure moderée. Il n'est pas dit qu'aux faibles profondeurs la résistance à la compression soit surpassée.

Aux Etats Unis, où le charbon est plus dur et moins fissuré que chez nous l'exploitation par chambres et piliers est très en vogue. Le charbon n'étant pas écrasé par le resserrement entre toit et mur doit être scié par des machines haveuses et détaché à coups de tir.

Notre calcul donne une idée de ce que serait la distribution de la pression si l'on compare le pilier de charbon avec un tampon de caoutchouc séparant toit et mur. Nous avons considéré les surfaces de glissement entre le charbon qui se dilate beaucoup et le toit et le mur qui n'expendent pas latéralement, comme parfaitement lisses et la seule remarque qu'on doit faire est qu'ainsi nous avons négligé le frottement. Plus loin nous verrons qu'un peu de frottement change complètement la répartition de la pression, dans ce sens, que la charge au ras du front diminue et augmente vers l'intérieur du pilier.

§ 5. *La pression du toit sur la couche de charbon, cette matière considérée comme parfaitement plastique.*

Quand on exploite la veine de charbon, quand on met à nu le front et le soulage de la pression, le charbon se détend vers le vide. Le volume de charbon resserré entre toit et mur qui s'approchent, ne change pas. Le front se déplace. Nous verrons que le frottement qui s'oppose à l'avancement de la masse, excite une pression latérale, qui s'accroît vers l'intérieur jusqu'à ce que le charbon soumis à la compression dans tous les sens, et à raison de $m = 2$, de volume constant, résiste à la compression comme de la matière absolument dure.

Pour exprimer le problème posé en formules et en nombres nous l'abordons par la théorie de la plasticité. Nous considérons le charbon comme de la matière parfaitement plastique, se comportant de même que l'argile, le plomb, le fer chauffé au rouge, et comprimé entre des surfaces parallèles où les tensions se disposent en vertu des lois de l'elasticité. Les calculs sont donnés dans l'annexe 3. Ils nous permettent d'évaluer en chiffres les tensions dans tout le massif du toit et du mur.

Pour un cas spécial nous calculons la distribution des tensions principales selon l'axe des Y et l'axe des Z.

Profondeur au dessous de la surface $h = 800$ m.

Poids spécifique des terrains $\gamma = 2.5$.

Epaisseur de la veine de charbon $2a = 1$ m.

Largeur du vide créé dans le charbon $2b = 4$ m.

Tension maximum de cisaillement (condition de plasticité) τ max. $= p_s = 10$ kg/cm^2.

Alors on obtient $c - b = 10$ m.

Le charbon se déforme, coule, plastiquement sur une distance de 10 m de chaque coté du creux (figure 15).

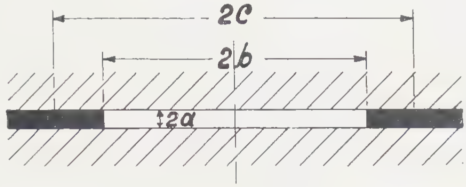

Fig. 15. La matière se comporte comme plastique de $2b$ jusqu'à $2c$ et en dehors de cette limite se comporte comme élastique. Nous pensons à la flexion légère du toit et du mur.

Les tensions (toutes des pressions) sont représentées à échelle dans la figure 16 pour tous les points le long de l'axe des Y et de l'axe des Z.

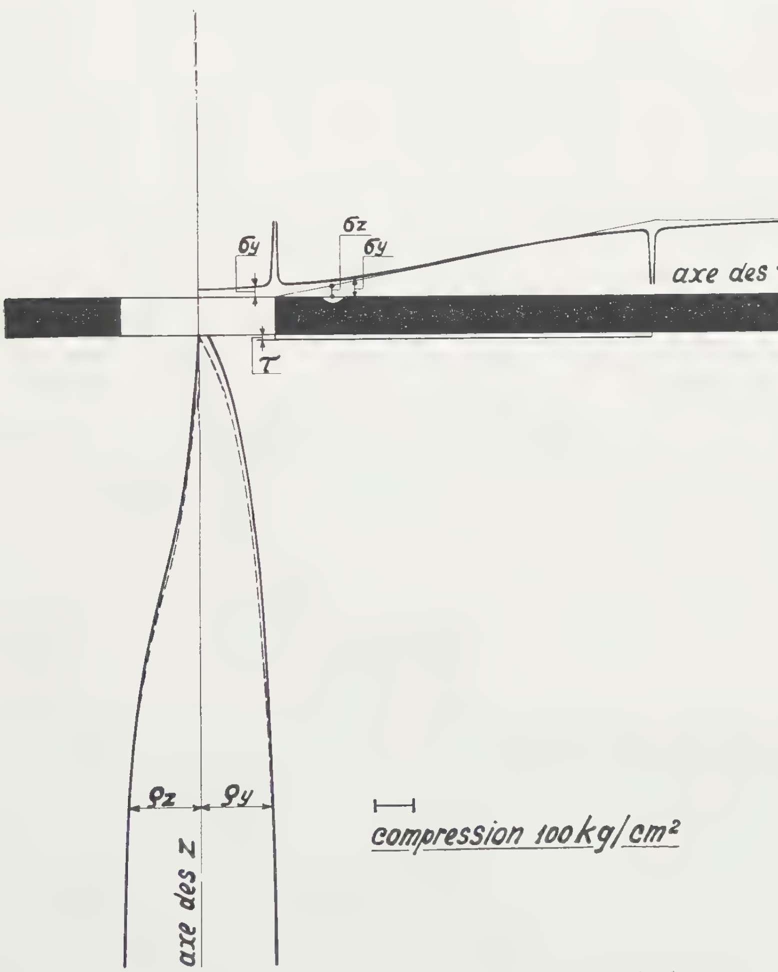

Fig. 16. Distribution des tensions normales σ_y et σ_z et de la tension de cisaillement τ le long de l'axe des Y et des tensions principales φ_y et φ_z le long de l'axe des Z pour de la matière plastique resserrée entre toit et mur de matière dure.

On remarque une inadmissibilité dans le cours des σ_z selon l'axe des Y. Ces tensions ne peuvent devenir infiniment grandes pour $y = b$ et $y = c$. Cette absurdité est due à une acceptation qu'on a du faire pour pouvoir intégrer les équations des tensions. L'influence de cette acceptation est discutée dans l'annexe 3. En vérité l'anomalie de σ_y dans les points $y = b$ et $y = c$ n'existe pas. On peut raccorder les trois parties de la courbe par des arrondissements légers.

La base de tout le calcul est que la tension de cisaillement τmax. $= p_s$ est constante (nous prenions $p_s = 10$ kg/cm^2) sur toute l'étendue des surfaces de contact entre toit et mur. Nous supposons qu'il existe la plus grande résistance possible contre le glissement de la masse plastique, considérons ces surfaces entaillées comme celles d'une lime.

Il résulte de ce calcul, de la représentation des résultats en la figure 16 que quand le toit et le mur resserrent une masse plastique ils sont épargnés. Au bord du front les tensions sont presque nulles et les tensions (pressions) élevées ne se développent qu'à une certaine distance du vide, là où elles s'allient avec la pression dans tous les sens de manière à devenir inoffensives.

Si le toit et le mur étaient séparés par de la matière vraiment plastique, on n'aurait besoin d'aucun soutènement, du moins si le rocher n'est pas trop fissuré et disloqué par les poussées tectoniques.

§ 6. *La pression du toit et du mur sur la couche de charbon, cette matière considérée comme pulvérulente.*

Mais le charbon n'est pas une matière parfaitement plastique. Il est au contraire très fragile et près du front il est fissuré par les plans de clivage. Le charbon ne devient plastique qu'à des pressions de quelques milliers d'atmosphères. C'est seulement à ces pressions qu'on peut en fabriquer des briquettes sans agglutinant. A 3000 atmosphères même on n'obtient qu'une quasi cohérence, les briquettes ainsi obtenues se désagrègent quand on les met dans l'alcool.

A la pression rencontrée dans nos exploitations, au plus quelques centaines d'atmosphères, le charbon écrasé, même disloqué en gros morceaux se comporte comme de la matière incohérente. Mieux que d'appliquer les lois de la plasticité on fera appel à la théorie de la poussée des terres, on appliquera les lois de l'équilibre dans les massifs à frottement interne, ce que nous faisons dans l'annexe 4.

Si l'on remplace le monolithe de charbon par du menu charbon, par du sable, il est possible de calculer par des méthodes connues, la distribution des pressions dans le rocher et dans la masse pulvérulente, quand on comprime celle-ci, jusqu'à ce que le front s'avance. On trouve alors que la matière s'écoule d'elle-même si l'on n'exerce pas une certaine pression sur le front. Mais si on prend cette précaution, si l'on soutient le charbon par du boisage on trouve que le frottement le long du mur et du toit

fait accroître la pression vers l'intérieur selon une loi exponentielle. Pour l'exemple calculé dans l'annexe 4 cet accroissement est représenté dans la figure 17. Nous avons construit notre figure avec une pression p_0 minime et avec un petit coëfficient de frottement. En réalité, l'accroissement de la pression, quand on s'éloigne du front et entre dans la couche, est tellement impétueuse qu'un peu de pression contre le front, qu'un peu de cohérance dans la matière suffit à faire s'élever la pression du toit et du mur très rapidement à des valeurs extrèmes.

Ainsi le frottement est capable de réaliser la distribution des pressions dans le rocher, représenté dans la figure 1, sauf la pression infiniment grande au ras du front du charbon. Celle-ci sera beaucoup atténuée.

Mais quand la pression du toit et du mur atteignent des valeurs tellement élevées, quand grâce au frottement il se développe dans le toit et le mur enserrant la matière incohérente une distribution de pressions comparable

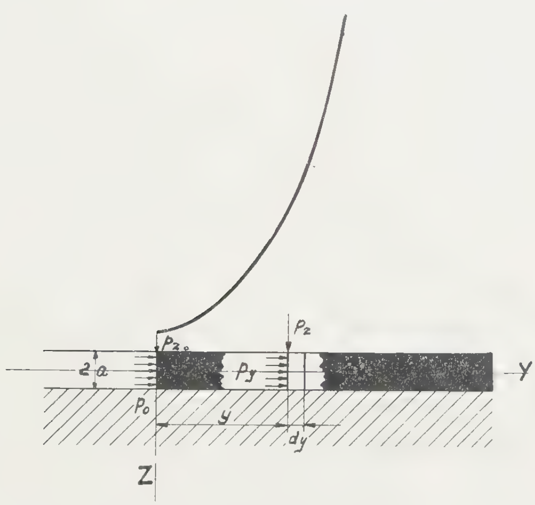

Fig. 17. Accroissement de la pression du toit et du mur sur de la matière incohérente selon une loi exponentielle et notations pour le calcul dans l'annexe 4.

à celle qui se produit sur une couche de matière parfaitement dure et indéformable, quand les pressions et surtout les tensions de cisaillement dans le toit et le mur surpassent de beaucoup la résistance à la rupture, il n'est plus permis de considérer le rocher comme incassable.

Nos calculs, basés sur les lois de l'élasticité, perdent alors leur validité.

Dans nos exploitations on observe des déplacements du front de charbon, fissuré, clivé, brisé en gros morceaux. Les géomètres des Mines de l'Etat ont fait maintes observations et relevés de ces déplacements, tant des avancements du front de charbon relatifs au toit et au mur que des translations absolues. Les déplacements relatifs, irréguliers sont de l'ordre d'une dizaine de cm pour une couche d'un mètre d'épaisseur. Dans la litérature on

trouve un grand nombre d'observations qui confirment nos constatations [1]).

Mais le charbon en site n'est pas précisement de la matière incohérente. Aussi quand on l'abat, le front n'est pas muni d'un boisage. Le fait qu'il ne s'écoule pas de lui même, qu'il faut l'abattre au marteau-pic pneumatique qu'on introduit dans les plans de clivage, démontre que nous n'avons pas encore la loi exacte qui commande les déplacements à l'intérieur de la couche de charbon exprimé vers le vide. En effet toute la masse n'est pas pulvérulente. Il y a de gros morceaux qui se réfractent. Pour les matières plastiques la résistance à la traction et à la compression sont égales, mais le charbon ne supporte que très peu de traction et la résistance à la compression est grande quand on ajoute de la pression dans tous les sens. Dans l'annexe 4 nous appliquons à la fin la loi, la condition de rupture pour cette espèce de matière et trouvons de nouveau la distribution, l'accroissement rapide de la pression, selon la formule exponentielle. Ce que nous avons dit de l'influence du frottement qui fait croître la pression près du front outre mesure, s'applique à plus forte raison pour le charbon encore cohérent, sauf que le front n'a même pas besoin de soutènement.

Par ce long chemin nous sommes arrivés à la conclusion qu'aux profondeurs où nous exploitons le charbon l'accroissement des pressions près du front est tel que le toit et le mur ne peuvent pas rester indemnes. Il s'y produit au moins des crevasses, mais si l'on examine le toit avec attention on trouve des dérangements de blocs qui certainement sont dus à l'abatage du charbon.

Jusqu'à présent nous avons toujours supposé que le rocher demeurait intact, mais de cette manière nous ne pouvions pas expliquer le comportement du chantier pendant l'exploitation. Nous espérons traiter dans un troisième chapitre la distribution des tensions dans le rocher et dans la veine autour de la taille quand on aura dépassé la résistance des deux matières à la rupture.

[1]) WEISSNER, Gebirgsbewegungen beim Abbau flachgelagerter Steinkohlenflösse, Glückauf 22 Okt. 1932, p. 945.

LÖFFLER, Zur Abbaudynamik bei streichendem Blindortbetrieb. Der Bergbau 9 Juni 1938.

Hydrodynamics. — *On the application of viscosity data to the determination of the shape of protein molecules in solution.* By J. M. BURGERS. (Mededeeling N⁰. 38 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft.)

(Communicated at the meeting of February 24, 1940.)

1. In Ch. III of the "Second Report on Viscosity and Plasticity" formulae have been given for the resistance of small particles of elongated form, and for their influence upon the effective viscosity of the liquid in which they are suspended; and the application has been discussed of these formulae to the results obtained with suspensions of polystyrenes by STAUDINGER and SIGNER [1]). A discussion of their application to suspensions of methyl cellulose has been given by SIGNER and V. TAVEL [2]). POLSON has applied the formula for the influence of such particles upon the effective viscosity to the analysis of data obtained with suspensions of proteins [3]), and a report of this work recently has been given by PEDERSEN in Part I, Ch. B, of SVEDBERG and PEDERSEN's new book "The Ultracentrifuge" [4]).

The way in which the formulae for the viscosity and for the frictional coefficient have been applied by POLSON and the discrepancy which has been found between certain calculated and observed values, calls for some remarks which will be collected in the following pages. In connection with these remarks a few data also will be supplied for some cases not treated in the "Second Report", *viz.* for disk-shaped particles (oblate rotational ellipsoids), and for certain systems consisting of rigidly connected spheres.

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1) J. M. BURGERS, Ch. III of the "Second Report on Viscosity and Plasticity", Verhand. Kon. Nederl. Akad. v. Wetenschappen (1e sectie) Vol. **16**, No. 4 (Amsterdam 1938), pp. 122—126 (resistance formulae), 145—153 (influence upon the effective viscosity), 168—181 (application to polystyrenes).

2) R. SIGNER und P. V. TAVEL, Die Form und Grösse von Methylcellulose-Molekeln in Lösung, Helv. Chim. Acta **21**, 535 (1938). The subject also has been treated in the chapter contributed by R. SIGNER to TH. SVEDBERG and KAI O. PEDERSEN's book "The Ultracentrifuge" (Oxford 1940), pp. 431—442, whereas cellulose acetates and some other linear high polymers are considered by E. O. KRAEMER and J. B. NICHOLS, *ibid.* pp. 416—431. — See also footnote 20) below.

3) A. POLSON, Ueber die Berechnung der Gestalt von Proteinmolekülen, Kolloid-Zeitschr. **88**, 51 (1939). — See also: Nature **137**, 740 (1936).

4) TH. SVEDBERG and KAI O. PEDERSEN, The Ultracentrifuge (Oxford 1940), pp. 38—44. The results of the calculations also have been given by TH. SVEDBERG in the "Opening Address to a discussion on the protein molecule", Proc. Roy. Soc. (London) B **127**, 9—10 (1939). — It may be mentioned that the problem of the determination of the shape of tobacco mosaic virus particles in solution has been discussed by J. R. ROBINSON, Nature **143**, 923 (1939).

The procedure followed by POLSON in order to obtain the molecular weight of a protein exclusively from viscosity and diffusion data, without having recourse to sedimentation measurements, was as follows: The length-diameter ratio L/d of the molecules, which were assumed to have the form of elongated rotational ellipsoids, was deduced from the specific increase of the viscosity of a solution, $\eta_{sp} = \eta/\eta_0 - 1$, making use of the known value of the partial specific volume V of the dissolved protein. The result was applied to derive the absolute dimensions of the molecule from its experimentally determined frictional constant f_{exp}, which can be obtained from the diffusion constant D (measured by means of an optical method [5]) through the equation $f_{exp} = RT/D$. From the dimensions and V the molecular weight then can be calculated.

However, on comparing the molecular weight found in this way with that deduced from observations on the sedimentation equilibrium, as is done *e.g.* in table 5 of "The Ultracentrifuge" [6]), it appears that the calculated values are much too low: by about 30 % when KUHN's formula for η_{sp} [7]) is used, and by about 50 % when the more exact formula, derived from JEFFERY's calculations by BURGERS [8]), is applied. A correct result could be obtained only when instead of these theoretical formulae, an empirical expression, given by POLSON [9]), is taken.

2. It would appear to the present writer that a more convenient basis for a discussion of the cause of the discrepancy can be obtained by arranging the calculations in a different way.

As explained in the "Second Report on Viscosity and Plasticity" [10]), molecular weights always should be deduced from direct measurements, in the present case preferably from the sedimentation equilibrium (M_e), or else from the sedimentation velocity in combination with the value of f_{exp} as obtained from diffusion measurements (M_s) [11]). These values of

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5) See A. POLSON, Nature **137**, 740 (1936), where it is stated that the diffusion constants were measured by the refractometric method of O. LAMM, Zeitschr. f. physik. Chem. A **138**, 313 (1928) and B **143**, 177 (1929).

6) "The Ultracentrifuge", Table 5, p. 44. — The same table occurs in TH. SVEDBERG, Proc. Roy. Soc. (London) B **127**, 10 (1939) and in A. POLSON, Kolloid-Zeitschr. **88**, 59 (1939).

7) W. KUHN, Zeitschr. f. physik. Chemie A **161**, 24 (1932).

8) Second Report, pp. 152—153.

9) This formula can be written in our notation as follows:

$$\eta_{sp}/c = V\,[4{,}0 + 0{,}098\,(L/d)^2],$$

as the quantity G used by POLSON is equal to cV. See A. POLSON, Kolloid-Zeitschr. **88**, 56 (1939); K. O. PEDERSEN, The Ultracentrifuge, p. 43.

10) Second Report, p. 184.

11) TH. SVEDBERG, The Ultracentrifuge, p. 9. — The fact that the two methods, in those cases where both can be applied, practically lead to the same values for the molecular weight, proves that diffusion and sedimentation velocity are both governed by the same mean frictional coefficient.

the molecular weights will be assumed as trustworthy here. When now the following assumptions are made: (1) that the volume taken in by the molecule in the solution can be deduced from M by means of the value measured for V; (2) that the molecule has the form of an elongated rotational ellipsoid, the value of η_{sp} can be used to find the value of L/d (under the circumstances of the viscosity experiment). The formula to be applied is:

$$\eta_{sp}/c = V \Lambda_{II} \quad \ldots\ldots\ldots \quad (1)$$

where η_{sp}/c is obtained from POLSON's data for $d(\eta/\eta_0)/dn$ [12]; V from data given in table 48, p. 406, of "The Ultracentrifuge"; while Λ_{II} is a function of L/d, tabulated in the "Second Report" [13].

Making use of the equation:

$$\pi L\, d^2/6 = M\, V/N_A \quad \ldots\ldots\ldots \quad (2)$$

(where N_A is AVOGADRO's number $= 6{,}06 \,.\, 10^{23}$), it then becomes possible to calculate the dimensions L and d of the molecule. These now can be used to find the frictional constants, f_1 for the motion in the direction of the axis of the molecule, and f_2 for the motion in the direction perpendicular to the axis; from these the mean frictional constant is derived by the equation: $1/f_m = 1/(3f_1) + 2/(3f_2)$. When we write:

$$f_m = 3\, \pi\, \eta\, L\, N_A/\lambda \quad \ldots\ldots\ldots \quad (3)$$

then for elongated rotational ellipsoids [14]):

$$\lambda = \frac{\log\,(L/d + \sqrt{L^2/d^2 - 1})}{\sqrt{1 - d^2/L^2}} \quad \ldots\ldots\ldots \quad (4)$$

(the *log* being the Napierian logarithm). Finally the sedimentation constant [15]) $S = (dx/dt)/\omega^2 x$ is given by:

$$S = \frac{1 - \varrho V}{3\, \pi\, \eta} \frac{M}{N_A\, L}\, \lambda \quad \ldots\ldots\ldots \quad (5)$$

When the calculations are executed in this way for the proteins mentioned by POLSON and PEDERSEN, and the results are compared with the experimental values of the sedimentation constant, it appears that the

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[12]) A. POLSON, Kolloid-Zeitschr. **88**, 58, Tab. VI (1939).

[13]) Second Report, p. 153.

[14]) The formulae for the resistance of an ellipsoid have been derived by OBERBECK, and are given *e.g.* in C. W. OSEEN, Hydrodynamik (Leipzig 1927), pp. 186—189. — See also: J. PERRIN, Journ. de Physique et le Radium (VII) **5**, 409 (1934) and **7**, 10—11 (1936).

[15]) See "The Ultracentrifuge", p. 5.

correct order of magnitude is obtained, but that on the average: $S_{obs} \cong = 1{,}3\ S_{calc}$, as will be seen from the following table: [16])

TABLE I.

(Elongated ellipsoids, unhydrated).

Name of the protein	M	$10^{24}\ \frac{M\,V}{N_A}$	$\frac{\eta_{sp}}{c\,V}$	$\frac{L}{d}$	$10^8 L$	$10^8 d$	$\frac{1-\varrho V}{3\pi\eta}$	$10^{13}.\ S_{calc}$	$10^{13}.\ S_{obs}$
Gliadin	27000	32200	14.55	20.9	300	14.3	2.94	1.64	2.1
Lactoglobulin	38000	47200	5.98	10.1	210	20.8	2.64	2.38	3.12
Ovalbumin	40500	50100	5.70	9.6	207	21.5	2.66	2.54	3.55
Haemoglobin	68000	84100	5.38	9.1	237	26.1	2.66	3.67	4.41
Serum albumin	68000	84000	6.52	11.0	269	24.5	2.67	3.44	4.46
Serum globulin	150000	184400	9.0	14.5	420	29.0	2.70	5.46	7.1
Amandin	330000	407000	7.04	11.8	476	40.4	2.69	9.8	12.5
Thyroglobulin	650000	770000	9.87	15.6	710	45.6	2.96	15.5	19.2
Homarus haemocyanin	800000	977000	6.39	10.8	600	55.7	2.75	18.7	22.6
Octopus ,,	2800000	3420000	9.03	14.5	1110	76.7	2.75	38.6	49.3
Helix pomatia ,,	6700000	8160000	6.36	10.7	1210	113	2.77	78	98.9

3. The discrepancy between calculated and observed results thus again turns up before us, but now in a form in which a better judgment can be made.

In order to explain this discrepancy SVEDBERG, PEDERSEN and POLSON have advanced the hypothesis that the protein molecules in the solution might be hydrated to such an extent, that their actual volume is equal to about 1,59 times the value $M\,V/N_A$ [17]). It is logical to assume that the consequent increase in dimensions then also must be taken into account in calculating the value of the frictional constant. It has been pointed out by KRAEMER that the hydration practically does not affect the driving force acting on a protein molecule in an aqueous solution during the

[16]) The data used in the calculations mostly have been taken from table 48, p. 406, of "The Ultracentrifuge". For the molecular weight the value of M_e has been taken, with the exception of that of *Octopus* haemocyanin, where M_s is used. The values of η_{sp}/c have been derived from POLSON's data, Kolloid-Zeitschr. **88**, 58, Tab. VI (1939); in the cases of serum globulin and *Helix pomatia* haemocyanin, however, the values of η_{sp}/cV have been derived from POLSON's Tab. III, *l.c.* p. 56, as it was not evident how the most suitable mean value should be obtained from the numbers given in Tab. VI.

[17]) See A. POLSON, Kolloid-Zeitschr. **88**, 56 (1939); K. O. PEDERSEN, The Ultracentrifuge, p. 43. — According to POLSON the so-called electroviscous effect can be neglected under suitably chosen conditions (see also PEDERSEN, The Ultracentrifuge, p. 26).

sedimentation process[18]); nor does it affect the number of molecules present in a solution of a given concentration of c grams of dry weight per unit of volume. Hence we can use the equation:

$$\frac{\eta_{sp}}{c} = \frac{N_A}{M} \frac{\pi L d^2}{6} \Lambda_{II} \quad \ldots \ldots \ldots \quad (6)$$

together with equation (5) also in the present case. Instead of eq. (2), we now, however, take:

$$\pi L d^2/6 = 1{,}59 \, . \, M V / N_A \quad \ldots \ldots \ldots \quad (7)$$

When the calculations are repeated in this way, a smaller value is obtained for L/d; L decreases slightly, whereas d increases, but the value of the sedimentation constant, as calculated from (5) with the new values of L and d, practically remains unchanged, so that the discrepancy is not removed in this way. The results have been given in Table II.

TABLE II.

(Elongated ellipsoids; volume increased by 59 % in consequence of hydration).

Name of the protein	$\frac{L}{d}$	$10^8\ L$	$10^8\ d$	$10^{13}\ S_{calc}$
Gliadin	14.7	271	18.8	1.63
Lactoglobulin	5.75	168	29.2	2.43
Ovalbumin	5.35	163	30.5	2.62
Haemoglobin	4.80	181	37.6	3.79
Serum albumin	6.55	222	33.9	3.51
Serum globulin	9.55	371	38.9	5.33
Amandin	7.2	400	55.5	9.9
Thyroglobulin	10.5	636	60.7	15.3
Homarus haemocyanin	6.4	495	77.4	18.9
Octopus ,,	9.6	985	103	38.3
Helix pomatia ,,	6.3	995	158	79

It is possible to arrange the calculations differently, making use neither of eq. (2), nor of eq. (7), but solving L and d directly from eqs. (6) and (5). The results arrived at in this way, however, appear to be extremely unprobable, as they lead to excessive values of L/d and to very low values of d, whereas the volume turns out even smaller than $M V/N_A$. As examples may be mentioned:

gliadin: $L/d = 360$; $L = 410 \, . \, 10^{-8}$; $d = 1{,}14 \, . \, 10^{-8}$
ovalbumin: $L/d = 710$; $L = 360 \, . \, 10^{-8}$; $d = 0{,}51 \, . \, 10^{-8}$
amandin: $L/d = 265$; $L = 730 \, . \, 10^{-8}$; $d = 2{,}76 \, . \, 10^{-8}$

[18]) E. O. KRAEMER, The Ultracentrifuge, p. 63.

It must be concluded, therefore, that the hypothesis of a large increase of volume in consequence of hydration does not help to remove the discrepancy between the results of the sedimentation and the viscosity measurements, so long as the assumption of an elongated ellipsoidal form is retained.

4. When the values of the ratio L/d for the proteins are calculated from the observed values of the sedimentation constant S, as has been done by POLSON [19]), results are obtained which are more than 50 % smaller than those found from η_{sp}, as given in table I above. It might be suggested, therefore, as a possibility for an explanation of the discrepancy, that the molecules should have different shapes in the two types of experiments to which they are subjected.

This possibility has been considered in the "Second Report" in connection with the application of the formulae to the results obtained with suspensions of polystyrenes, where a similar discrepancy between the calculated and the observed values of the sedimentation constant was found [20]). It was shown there that the forces exerted by the liquid upon a

19) A. POLSON, Kolloid-Zeitschr. **88**, 56, Tab. III (1939).

20) Second Report, pp. 176—178. The differences between the calculated and the observed values of the sedimentation constant S in the case of polystyrenes is seen from the table given at p. 176, where it must be kept in mind that the calculated values refer to the motion in the direction of the axis only, and consequently must be divided by about 1,5 in order to give the mean values for all directions of the axis in space. It will be observed that in the case of the polystyrene with $M = 270000$ the discrepancy is much greater than those which are found for the proteins. In the case of the suspensions of methyl cellulose, on the other hand, which were investigated by SIGNER and V. TAVEL (see footnote 2) above), the discrepancy between the calculated and the observed sedimentation velocities is much smaller. In "The Ultracentrifuge", p. 437, Table 54, the values of V and η_{sp}/c have been given; the values of L/d and of L and d calculated from these are collected in Table 55 at the same page. For the calculation of the sedimentation constant SIGNER has used a formula somewhat differing from the one applied in the present text (see Helv. Chim. Acta **21**, 542 (1938) and "The Ultracentrifuge", p. 435); when the values are re-calculated with the aid of eqs. (5) and (4) given above, the following results are obtained:

Fraction	IV	III	II
Molecular weight (M_w)	14100	24300	38100
L/d	77	109	139
$10^{13}\ S_{calc}$	0.70	0.73	0.85
$10^{13}\ S_{obs}$	0.83	0.79	0.89

Here again the calculated values are less than the observed ones, although the difference is becoming smaller for the higher molecular weights. SIGNER remarks that the lengths of the molecules, calculated from η_{sp}, exceed the lengths calculated from the number of

molecule in consequence of the shearing motion which exists in a viscosity experiment, tend to produce an elongation of the molecule; on the other hand, during a sedimentation experiment there is practically no tendency towards a deformation of the molecule. Nevertheless, the fact that the viscosity of a protein solution does not appear to be markedly dependent upon the velocity gradient [21]), makes it unprobable that large deformations are caused by the shearing motion of the liquid, unless it might be supposed that the molecule should possess two relatively stable forms, one of which would appear during the sedimentation experiments, while the other would appear in the viscosity experiments, even if the shearing velocity would be small.

5. There remains, however, an important point which needs consideration, *viz.* that the formulae used in the calculations have been deduced for particles of *ellipsoidal* shape, and that numerically different results must be expected for particles having other shapes. Unhappily there do not exist exact formulae for particles of other shapes; the formulae given in the "Second Report" for cylindrical particles are approximations, which, although useful for great values of L/d, are not sufficiently precise for application in the problem here before us.

It might be supposed that the molecules of the proteins should have the form of *oblate rotational ellipsoids*; for particles of such forms exact expressions can be given. Equations (1), (2) and (5) can be used also in this case; the values of Λ_{II} and λ then must be calculated anew. When it is assumed that the Brownian movement is sufficiently strong to make all directions of the axis in space equally probable, then for Λ_{II} we can start from eq. (14. 17) of the "Second Report" (p. 151), provided for C_1, C_2, C_3 we now substitute expressions which are valid for oblate ellipsoids of revolution, and which also have been given by JEFFERY [22]), like those for the elongated ones. A few results are collected in the following table, which is a counterpart to the table given at p. 153 of the "Second Report" for elongated ellipsoids (as before, we have written $d = 2b$ for the equatorial diameter, and $L = 2a$ for the axial diameter of the ellipsoids):

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glucose residues present in the molecule. — In the case of the cellulose acetates, considered by KRAEMER and NICHOLS ("The Ultracentrifuge", pp. 426—431), the opposite relation is found: here the lengths calculated from η_{sp}/cV remain below the maximum lengths deduced from the number of structural units, while the calculated sedimentation velocities exceed the observed ones. For other linear high polymers the calculated values of S again are too low; it is mentioned that here the molecules may be coiled, so that similar deformations might be possible as in the case of the polystyrenes.

[21]) A. POLSON, Kolloid-Zeitschr. **88**, 57, (1939).

[22]) G. B. JEFFERY, The motion of ellipsoidal particles immersed in a viscous fluid, Proc. Roy. Soc. (London) A **102**, 174—175 (1922—1923). The expressions for C_1, C_2, C_3 can be derived from eq. (61), p. 174, in which the values given at p. 175, eqs. (68), have to be substituted for α_0', β_0', β_0'' ($\alpha_0'' = b^2\alpha_0' - \beta_0''/2$, according to p. 173).

Values of A_{II} for oblate rotational ellipsoids.

$b/a = d/L$	A_{II}	$b/a = d/L$	A_{II}
1	2.50	12	6.39
5	3.56	15	7.64
6	3.95	20	9.74
8	4.74	25	11.85
10	5.56	30	13.96

From $b/a = 35$ onward the approximate formula

$$A_{II} \simeq 4b/3\,\pi a + 1{,}19$$

is sufficiently accurate.

When the frictional constant in the present case is written in the form:

$$f_m = 3\,\pi\,\eta\,d\,N_A/\lambda \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

and the formula for the sedimentation constant as:

$$S = \frac{1-\varrho V}{3\,\pi\,\eta}\,\frac{M}{N_A\,d}\,\lambda \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

then the expression for λ becomes, for oblate ellipsoids of revolution

$$\lambda = \frac{\operatorname{arc\,tg}\sqrt{d^2/L^2-1}}{\sqrt{1-L^2/d^2}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

When the calculations are performed in the same way as was indicated in section 2, again assuming that the volume of a molecule is unaffected by hydration, the curious result is obtained that practically the same values of the sedimentation constant are found as had been derived upon the assumption of an elongated ellipsoidal shape. This will be seen from Table III.

TABLE III.
(Oblate ellipsoids, unhydrated).

Name of the protein	$\frac{d}{L}$	$10^8\,d$	$10^8\,L$	$10^{13}\,S_{\text{calc.}}$
Gliadin	31.4	125	3.97	1.62
Lactoglobulin	11.0	100	9.1	2.47
Ovalbumin	10.3	100	9.6	2.65
Haemoglobin	9.6	116	12.1	3.73
Serum albumin	12.3	125	10.2	3.57
Serum globulin	18.3	186	10.2	5.43
Amandin	13.55	219	16.2	10.0
Thyroglobulin	20.3	310	15.3	15.5
Homarus haemocyanin	12.0	282	23.5	19.2
Octopus ,,	18.3	493	27.0	38.9
Helix pomatia ,,	11.9	570	48.0	79

One still might wish to repeat the calculations for a different value of the volume of a molecule, in order to see whether the assumption of hydration might help us now. It could be attempted again to calculate L and d directly from eqs. (6) and (5). It is found, however, that upon the supposition of an oblate ellipsoidal form no solution of these equations can be obtained. By way of example in the following table the cases of gliadin and amandin have been considered. Starting with the value of $L\,d^2\,\Lambda_{II} = (6/\pi)\,.\,(\eta_{sp}/c)\,.\,(M/N_A)$ various assumptions are tried for the ratio d/L; for each of these values the corresponding values of L and d can be found, and the calculated value of L/λ can be compared with the experimental value of this quantity, which is equal to $(1-\varrho V)/(3\pi\eta)\,.\,(M/N_A)\,.\,(1/S_{obs})$.

		gliadin	*amandin*
experimental value of $L\,d^2\,\Lambda_{II}$:		891000.10^{-24}	5450000.10^{-24}
,, ,, ,, L/λ :		$62{,}4.10^{-8}$	117.10^{-8}
oblate ellipsoids			
value assumed for d/L		calculated value of L/λ	
	400	81.10^{-8}	149.10^{-8}
	100	81.10^{-8}	148.10^{-8}
	25	81.10^{-8}	147.10^{-8}
	5	77.10^{-8}	141.10^{-8}
sphere	1	71.10^{-8}	130.10^{-8}
elongated ellipsoids			
value assumed for L/d			
	2	73.10^{-8}	134.10^{-8}
	10	82.10^{-8}	150.10^{-8}
	100	70.10^{-8}	128.10^{-8}
	200	$65{,}6.10^{-8}$	$119{,}5.10^{-8}$
	400	$61{,}5.10^{-8}$	$112{,}4.10^{-8}$

Hence it is found that the only simultaneous solutions of the equations are obtained for elongated ellipsoids with L/d between 200 and 400, which solutions already have been given at the end of section **3**.

(To be continued.)

Mathematics. — *Ueber lineare Linienkomplexe bei vier Geraden im R_4.* Von R. WEITZENBÖCK.

(Communicated at the meeting of February 24, 1940.)

Uebersicht. Ich bestimme im Folgenden den allgemeinsten linearen Linienkomplex, der kovariant zu vier Geraden in vierdimensionalen Raume gehört und beweise, dass durch vier Geraden allgemeiner Lage und ihre vier Transversalen, also durch eine sogenannte „Doppelvier" ∞^2 lineare Komplexe gehen, unter denen drei spezielle vorhanden sind. Die Brennpunkte der übrigen füllen eine Ebene aus.

§ 1.

Ich habe vor einiger Zeit bewiesen[1]) dass sich jede ganze rationale projektive Invariante von fünf Geraden a, α, p, m und π im R_4, die wir auch durch die Ziffern 1, 2, 3, 4 und 5 andeuten, ganz und rational ausdrücken lassen durch die Typen

$$A_{1,23,45} = (12^2\,3^2)(14^2\,5^2) = 2^4 \cdot \sum_{ik} a_{ik}\,(S'_{23}\,S'_{45})_{ik} \quad . \; . \; . \quad (1)$$

wobei z.B. S'_{23} der Verbindungsraum der Geraden 2 und 3 ist. $A_{1,23,45}=0$ drückt also aus, dass die Schnittebene der beiden dreidimensionalen Räume S'_{23} und S'_{45} von der Geraden 1 geschnitten wird. Die Invarianten

$$A_{1,23,45} = -A_{1,45,23}$$

sind zyklisch-symmetrisch in 123 und ebenso in 145.

Die Frage nach dem allgemeinsten linearen Strahlenkomplex K, der zu vier Geraden 1 bis 4 kovariant ist, lautet dann in algebraischer Formulierung: es ist die allgemeinste projektive Invariante K der fünf Geraden 1 bis $5=\pi$ zu ermitteln, die in den π_{ik} linear ist. Nach Obigem ist demnach K linear aufzubauen aus Invarianten $A_{m,ik,rs}$, worin genau einer der Indizes ein π ist mit Koeffizienten, die Polynome der $A_{m,ik,rs}$ sind, wo alle fünf Indizes aus 1 bis 4 genommen sind. Von diesen zuletzt genannten Invarianten habe ich bewiesen, dass sie auf vier algebraisch-unabhängige zu reduzieren sind, z.B. auf

$$\left.\begin{aligned} A_1 &= A_{4,13,12} \\ A_2 &= A_{1,23,24} \\ A_3 &= A_{2,13,34} \\ A_4 &= A_{3,14,24} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

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[1]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 245—252 (1939).

Die Invarianten A mit einer Reihe π verteilen wir auf zwei Klassen: erstens Invarianten $J_n = A_{m,ik,rs}$ in der alle fünf Geraden vorkommen. Solche J_n gibt es 15, die sich aber vermöge der zyklischen Symmetrie linear, mit konstanten Koeffizienten durch sechs linear-unabhängige ausdrücken lassen [2]), z.B. durch die folgenden sechs:

$$\left.\begin{array}{ll} J_1 = A_{1,23,4\pi} & J_4 = A_{2,13,4\pi} \\ J_2 = A_{1,24,3\pi} & J_5 = A_{2,14,3\pi} \\ J_3 = -A_{1,34,2\pi} & J_6 = -A_{2,34,1\pi} \end{array}\right\} \quad \ldots\ldots \quad (3)$$

Zur zweiten Klasse rechnen wir diejenigen Invarianten $K_{rs} = A$, in denen eine der Geraden 1 bis 4 fehlt. Macht man wieder von der zyklischen Symmetrie Gebrauch, so gibt es deren zwölf, nämlich:

$$\left.\begin{array}{llll} K_{13} = A_{2,34,3\pi} & K_{24} = A_{3,14,4\pi} & K_{31} = A_{4,12,1\pi} & K_{42} = A_{1,23,2\pi} \\ K_{14} = A_{3,24,4\pi} & K_{21} = A_{4,13,1\pi} & K_{32} = A_{1,24,2\pi} & K_{43} = A_{2,13,3\pi} \\ K_{12} = A_{4,23,2\pi} & K_{23} = A_{1,34,3\pi} & K_{34} = A_{2,14,4\pi} & K_{41} = A_{3,12,1\pi} \end{array}\right\} \quad . \quad (4)$$

Hier gibt der erste Index r von K_{rs} die Gerade an, die nicht in A vorkommt; die zweite Index s gibt die Gerade s, durch welche die Ebene mit der Gleichung $K_{rs} = 0$ geht. $K_{rs} = 0$ ist also die Ebene der beiden Geraden s und r^*. Dabei ist z.B. 4^* die Transversale von 1, 2 und 3.

Die Invarianten der Gestalt (4) mit π als erstem Index sind durch die K_{rs} von (4) ausdrückbar. So ist z.B. wegen der zyklischen Symmetrie

$$A_{\pi,23,24} = -A_{4,23,2\pi} - A_{2,23,4\pi} = -K_{12},$$

da $A_{2,23,4\pi}$ verschwindet.

Das allgemeinste K hat demnach die Gestalt

$$K = \sum_1^6 \mu_i J_i + \sum_{rs} \mu_{rs} K_{rs}, \quad \ldots\ldots\ldots \quad (5)$$

wobei die Koeffizienten μ_i und μ_{rs} Polynome der vier Invarianten A_i von (2) sind.

§ 2.

Wir beweisen jetzt, dass die Invarianten (3) auf die K_{rs} von (4) reduzierbar sind.

Es sei b mit $a = 1$ äquivalent; dann haben wir

$$P = A_{a,23,24} A_{b,34,3\pi} = (a\,2^2\,3^2)(a\,2^2\,4^2)(b\,3^2\,4^2)(b\,3^2\,\pi^2).$$

Hier bringen wir die Reihe b des dritten Faktors in den ersten:

$$P = (b\,2^2\,3^3)(a\,3^2\,4^2)(a\,2^2\,4^2)(b\,3\;\;\pi^2) + 2\,(ab\,23^2)(a\,2^2\,4^2)(23^2\,4^2)(b\,3^2\pi^2).$$

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[2]) l.c., p. 251.

Der erste Term rechts ist hier $A_{1,34,24} \cdot A_{1,23,3\pi}$; beim zweiten vereinigen wir die beiden Reihen a im ersten Klammerfaktor und erhalten

$$2\,(ab\,23^2)\,(a\,2^2\,4^2) = (a^2\,23^2)\,(b\,2^2\,4^2),$$

da $(a^2\,b\,..)$ zu Null führt. Somit kommt, da

$$A_{2,a3,34} = -A_{1,23,34}$$

ist

$$A_{1,23,24}\,A_{1,34,3\pi} + A_{1,24,34}\,A_{1,23,3\pi} = -A_{1,23,34}\,A_{1,24,3\pi} \quad . \; . \quad (6)$$

oder, in den A_i und K_{rs} ausgedrückt:

$$A_{1,24,3\pi} = K_{23}\frac{A_2}{A_3} - K_{43}\frac{A_4}{A_3}. \quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

Durch Permutation der Ziffern 1, 2, 3, 4 ergibt sich hieraus für alle Invarianten (3) Reduktion auf die K_{rs} von (4).

Es ist für das Folgende nützlich die Wirkung der sechs Transpositionen $i \times k$ der vier Ziffern 1, 2, 3, 4 auf die Invarianten A_i und K_{rs} zu übersehen. Wir stellen dies in einer Tabelle zusammen, wobei wir links die Invarianten und oben die Transpositionen als Eingangsreihen setzen. Bei den K_{rs} geben wir in der Tabelle nur das Indexpaar, lassen also den Buchstaben K weg. So findet man also z.B. in der mit K_{43} beginnenden Zeile unter der Transposition 2×3 in der Tabelle -42 stehen. Das heisst: vertauscht man in K_{43} die zweite mit der dritten Geraden, so entsteht $-K_{42}$.

	1 2	1 3	1 4	2 3	2 4	3 4
A_1	$-A_2$	$-A_3$	$-A_4$	$-A_1$	$-A_1$	$-A_1$
A_2	$-A_1$	$-A_2$	$-A_2$	$-A_3$	$-A_4$	$-A_2$
A_3	$-A_3$	$-A_1$	$-A_3$	$-A_2$	$-A_3$	$-A_4$
A_4	$-A_4$	$-A_4$	$-A_1$	$-A_4$	$-A_2$	$-A_3$
K_{13}	+23	−31	+43	−12	−13	−14
K_{14}	+24	−34	+41	−14	−12	−13
K_{12}	+21	−32	+42	−13	−14	−12
K_{24}	+14	−24	−21	+34	−42	−23
K_{21}	+12	−23	−24	+31	−41	−21
K_{23}	+13	−21	−23	+32	−43	−24
K_{31}	−32	−13	−34	+21	−31	+41
K_{32}	−31	−12	−32	+23	−34	+42
K_{34}	−34	−14	−31	+24	−32	+43
K_{42}	−41	−42	+12	−43	−24	+32
K_{43}	−43	−41	+13	−42	−23	+34
K_{41}	−42	−43	+14	−41	−21	+31

Durch Vertauschung von 1 mit 3 ergibt sich so aus (7)

$$A_{3,24,1\pi} = -K_{21}\frac{A_2}{A_1} + K_{41}\frac{A_4}{A_1}.$$

Addieren wir dies zu (7), so entsteht wegen

$$A_{\pi,24,13} = -A_{1,24,3\pi} - A_{3,24,1\pi}:$$

$$L_2 = A_{\pi,13,24} = K_{23}\frac{A_2}{A_3} - K_{43}\frac{A_4}{A_3} + K_{41}\frac{A_4}{A_1} - K_{21}\frac{A_2}{A_1} \quad . \; . \quad (8b)$$

und hieraus, wieder durch Vertauschungen:

$$L_1 = A_{\pi,12,34} = K_{23}\frac{A_2}{A_3} + K_{42}\frac{A_4}{A_2} - K_{41}\frac{A_4}{A_1} - K_{31}\frac{A_3}{A_1} \quad . \; . \quad (8a)$$

$$L_3 = A_{\pi,14,23} = -K_{24}\frac{A_2}{A_4} - K_{34}\frac{A_3}{A_4} + K_{31}\frac{A_3}{A_1} + K_{21}\frac{A_2}{A_1}. \quad . \quad (8c)$$

§ 3

Wir können jetzt im allgemeinen Ansatz (5) für den Komplex K die Invarianten J weglassen und

$$K = \sum_{rs} \lambda_{rs} K_{rs} = \lambda_{13} K_{13} + \ldots + \lambda_{41} K_{41} = 0 \quad . \; . \; . \; . \quad (9)$$

setzen.

Substituieren wir hierin der Reihe nach $\pi = 1, 2, 3$ und 4, drücken also die Bedingungen dafür aus, dass die Geraden 1 bis 4 dem Komplex K angehören, so ergeben sich die Gleichungen

$$\lambda_{13} A_3 + \lambda_{14} A_4 + \lambda_{12} A_2 = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (10_1)$$

$$\lambda_{24} A_4 + \lambda_{21} A_1 + \lambda_{23} A_3 = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (10_2)$$

$$\lambda_{31} A_1 + \lambda_{32} A_2 + \lambda_{34} A_4 = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (10_3)$$

$$\lambda_{42} A_2 + \lambda_{43} A_3 + \lambda_{41} A_1 = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (10_4)$$

Wir nennen wieder 1^* die Transversale der Geraden 2, 3 und 4. Dann lauten die Bedingungen, dass die Geraden 1^* bis 4^* den Komplex K angehören:

$$\frac{1}{A_2}\lambda_{21} - \frac{1}{A_3}\lambda_{31} + \frac{1}{A_4}\lambda_{41} = 0. \quad . \; . \; . \; . \; . \; . \quad (10_1^*)$$

$$\frac{1}{A_1}\lambda_{12} + \frac{1}{A_3}\lambda_{32} - \frac{1}{A_4}\lambda_{42} = 0. \quad . \; . \; . \; . \; . \; . \quad (10_2^*)$$

$$\frac{1}{A_2}\lambda_{23} - \frac{1}{A_1}\lambda_{13} + \frac{1}{A_4}\lambda_{43} = 0. \quad . \; . \; . \; . \; . \; . \quad (10_3^*)$$

$$\frac{1}{A_1}\lambda_{14} - \frac{1}{A_2}\lambda_{24} + \frac{1}{A_3}\lambda_{34} = 0. \quad . \; . \; . \; . \; . \; . \quad (10_4^*)$$

Diese acht Gleichungen (10) und (10*) sind, wie man leicht nachrechnet, durch eine lineare Abhängigkeit verbunden. Es ist nämlich, identisch in den zwölf λ_{rs}:

$$\left.\begin{aligned}\frac{1}{A_1}(10_1)+\frac{1}{A_2}(10_2)+\frac{1}{A_3}(10_3)+\frac{1}{A_4}(10_4)=\\ =A_1(10_1^*)-A_2(10_2^*)+A_3(10_3^*)-A_4(10_4^*).\end{aligned}\right\} \quad . \; . \; . \quad (11)$$

§ 4.

Die Gleichungen $K_{rs}=0$ stellen zwölf Ebenen des R_4 dar. Da zwischen elf Ebenen im R_4 stets wenigstens eine lineare Beziehung existiert, müssen zwischen den zwölf Komitanten (4) wenigstens zwei lineare, identisch in allen Veränderlichen π_{ik} geltende Beziehungen vorhanden sein. Diese wollen wir jetzt ermitteln.

Zunächst gelten für die λ_{rs} mit

$$K=\sum_{rs}\lambda_{rs}K_{rs}=0 \quad \{\text{in allen } \pi_{ik}\} \quad . \; . \; . \; . \; . \quad (12)$$

die Gleichungen (10): sie entstehen ja dadurch, dass wir $\pi_{ik}=1$ bis 4 und $=1^*$ bis 4^* setzen.

Der Schnittpunkt der beiden Geraden 1 und 4^* ist gegeben durch $a_i\,(a\,S'_{23})$, der der beiden Geraden 2 und 1^* durch $\alpha_i\,(\alpha\,S'_{34})$: die Verbindingslinie dieser beiden Punkte hat daher die Koordinaten

$$\pi_{ik}=(a\,\alpha)_{ik}\,(a\,S'_{23})\,(\alpha\,S'_{34}).$$

Setzt man dies in (12) ein, so erhält man

$$\lambda_{24}\,A_3\,A_4+\lambda_{23}\,A_3^2-\lambda_{34}\,A_2\,A_4=0.$$

Hier gibt die Vertauschung von 3 mit 4 nach der Tabelle

$$\lambda_{23}\,A_3\,A_4+\lambda_{24}\,A_4^2+\lambda_{43}\,A_2\,A_3=0.$$

Hieraus und aus (10_3^*) kann man λ_{43} eliminieren und erhält

$$\lambda_{24}=-\lambda_{13}\frac{A_2\,A_3}{A_1\,A_4}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

Die sechs Vertauschungen $i\times k$ führen nach der Tabelle zu fünf weiteren Gleichungen:

$$\left.\begin{aligned}&\lambda_{14}=-\lambda_{23}\frac{A_1\,A_3}{A_2\,A_4} \qquad && \lambda_{31}=\lambda_{13}\frac{A_3^2}{A_1^2}\\ &\lambda_{42}=-\lambda_{13}\frac{A_3\,A_4}{A_1\,A_2} && \lambda_{21}=\lambda_{43}\frac{A_2\,A_3}{A_1\,A_4}\\ &\lambda_{34}=\lambda_{12}\frac{A_2\,A_3}{A_1\,A_4}\end{aligned}\right\} \quad . \; . \; . \; . \quad (13)$$

Setzt man dies alles in die Gleichungen (10) ein, so findet man, dass sich auch die übrigen λ_{rs} durch λ_{13} und λ_{23} ausdrücken lassen; man erhält:

$$\left.\begin{array}{ll}
\lambda_{12} = -\lambda_{13}\dfrac{A_3}{A_2} + \lambda_{23}\dfrac{A_1 A_3}{A_2^2} & \lambda_{34} = -\lambda_{13}\dfrac{A_3^2}{A_1 A_4} + \lambda_{23}\dfrac{A_3^2}{A_2 A_4} \\[2ex]
\lambda_{21} = \lambda_{13}\dfrac{A_2 A_3}{A_1^2} - \lambda_{23}\dfrac{A_3}{A_1} & \lambda_{43} = \lambda_{13}\dfrac{A_4}{A_1} - \lambda_{23}\dfrac{A_4}{A_2} \\[2ex]
\lambda_{32} = -\lambda_{23}\dfrac{A_3^2}{A_2^2} & \lambda_{41} = \lambda_{23}\dfrac{A_3 A_4}{A_1 A_2}.
\end{array}\right\} \quad . \quad (13a)$$

Mit diesen Werten werden auch die Gleichungen (10*) identisch in λ_{13} und λ_{23} befriedigt.

Setzen wir jetzt diese λ_{rs} in (12) ein, so entsteht, wenn wir alle Nenner beseitigen:

$$K = A_1 A_2 A_3 A_4 (\lambda_{13} A_2 M_{13} - \lambda_{23} A_1 M_{23}) \equiv 0,$$

woraus

$$M_{13} \equiv 0 \quad \text{und} \quad M_{23} \equiv 0 \quad \{\text{in allen } \pi_{ik}\} \quad . \quad . \quad . \quad . \quad (14)$$

folgen. Hierbei sind die M_{rs} Ausdrücke, die sich mit Hilfe der Abkürzung

$$rs = K_{rs}\frac{A_r}{A_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

wie folgt darstellen lassen:

$$M_{13} = 21 - 12 + 13 + 31 - 24 - 42 + 43 - 34 \quad . \quad . \quad . \quad (16a)$$

$$M_{23} = 21 - 12 + 14 - 41 + 32 - 23 + 43 - 34 \quad . \quad . \quad . \quad (16b)$$

Bei den Transpositionen $i \times k$ bleibt M_{23} invariant oder geht in $-M_{13}$ oder in $M_{13} - M_{23}$ über.

Man beweist leicht, dass die zwei Identitäten (14) von einander unabhängig sind und dass alle linearen Abhängigkeiten der K_{rs} auf diese beiden Identitäten zurückführbar werden. Man kann also nach (16*a*) und (16*b*) z.B. K_{42} und K_{43} linear-homogen durch die zehn übrigen K_{rs} von (4) ausdrücken. Die allgemeinste, in den π_{ik} lineare Komitante lautet dann

$$K = \sum_{rs} \mu_{rs} K_{rs} = \mu_{13} K_{13} + \ldots + \mu_{34} K_{34} + \mu_{41} K_{41} \quad . \quad . \quad (17)$$

und es ist, in der Bezeichnung (15):

$$42 = 13 + 31 + 23 - 32 + 41 - 14 - 24 \quad . \quad . \quad . \quad . \quad (18)$$

$$43 = 12 - 21 + 41 - 14 + 23 - 32 + 34 \quad . \quad . \quad . \quad . \quad (19)$$

§ 5.

Wenn K von (17) durch die Geraden 1 bis 4 und 1* bis 4* gehen soll, gelten für die λ_{rs} die Gleichungen (10), wobei noch $\lambda_{42} = \lambda_{43} = 0$ genommen ist. Aus (10) folgt dann auch $\lambda_{41} = 0$ und die übrigen Gleichungen führen auf

$$\left.\begin{array}{ll}
\lambda_{13} A_3 + \lambda_{14} A_4 + \lambda_{12} A_2 = 0 & \dfrac{1}{A_2}\lambda_{21} - \dfrac{1}{A_3}\lambda_{31} = 0 \\[2ex]
\lambda_{24} A_4 + \lambda_{21} A_1 + \lambda_{23} A_3 = 0 & \dfrac{1}{A_1}\lambda_{12} + \dfrac{1}{A_3}\lambda_{32} = 0 \\[2ex]
\lambda_{31} A_1 + \lambda_{32} A_2 + \lambda_{34} A_4 = 0 & \dfrac{1}{A_2}\lambda_{23} - \dfrac{1}{A_1}\lambda_{13} = 0 \\[2ex]
\multicolumn{2}{c}{\dfrac{1}{A_1}\lambda_{14} - \dfrac{1}{A_2}\lambda_{24} + \dfrac{1}{A_3}\lambda_{34} = 0.}
\end{array}\right\} \quad . \quad . \quad (20)$$

Hieraus folgt, dass sich alle λ_{rs} durch drei unabhängige, z.B. λ_{12}, λ_{21} und λ_{13} ausdrücken lassen:

$$\begin{array}{llll}
* & \lambda_{24} = -\dfrac{A_1}{A_4}\lambda_{21} - \dfrac{A_2 A_3}{A_1 A_4}\lambda_{13} & \lambda_{31} = \dfrac{A_3}{A_2}\lambda_{21} & \lambda_{42} = 0 \\[2ex]
\lambda_{14} = -\dfrac{A_2}{A_4}\lambda_{12} - \dfrac{A_3}{A_4}\lambda_{13} & * & \lambda_{32} = -\dfrac{A_3}{A_1}\lambda_{12} & \lambda_{43} = 0 \\[2ex]
* & \lambda_{23} = \dfrac{A_2}{A_1}\lambda_{13} & \lambda_{34} = -\dfrac{A_1 A_3}{A_2 A_4}\lambda_{21} + \dfrac{A_2 A_3}{A_1 A_4}\lambda_{12} & \lambda_{41} = 0
\end{array}$$

Somit wird K:

$$\left.\begin{array}{l}
K = \lambda_{12}\left\{K_{12} - \dfrac{A_2}{A_4}K_{14} - \dfrac{A_3}{A_1}K_{32} + \dfrac{A_2 A_3}{A_1 A_4}K_{34}\right\} + \\[2ex]
\quad + \lambda_{21}\left\{-\dfrac{A_1}{A_4}K_{24} + \dfrac{A_3}{A_2}K_{31} - \dfrac{A_1 A_3}{A_2 A_4}K_{34} + K_{21}\right\} + \\[2ex]
\quad + \lambda_{13}\left\{K_{13} - \dfrac{A_3}{A_4}K_{14} - \dfrac{A_2 A_3}{A_1 A_4}K_{24} + \dfrac{A_2}{A_1}K_{23}\right\}.
\end{array}\right\} \quad . \quad . \quad (21)$$

Wenn wir in den Ausdrücken L_i der Gleichungen (8) vermöge (18) und (19) die 42 und 43 eliminieren, so entsteht

$$L_1 = 13 + 23 - 14 - 24, \quad L_2 = 14 + 32 - 34 - 12, \quad L_3 = 31 + 21 - 24 - 34 \quad (22)$$

und dies sind, bis auf Faktoren $\dfrac{A_i}{A_k}$ die Koeffizienten der λ_{rs} in (21). Wir erhalten somit den *Satz*:

Durch die acht Geraden 1 bis 4 und 1 bis 4* einer Doppelvier des R_4 gehen ∞^2 lineare Strahlenkomplexe. Sie sind gegeben durch die Schaar*

$$\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = \lambda_1 A_{\pi,12,34} + \lambda_2 A_{\pi,13,24} + \lambda_3 A_{\pi,14,23} = 0. \quad (23)$$

Hier ist z.B.

$$A_{\pi,12,34} = (\pi S'_{12})(\pi S'_{34}) = 0$$

ein spezieller Komplex, nämlich die Schnittebene der beiden Räume S'_{12} und S'_{34}.

Die ∞^2 Strahlenkomplexe (23) haben ∞^2 Brennpunkte, die auf vier Ebenen liegen. Drei davon sind die Ebenen $L_i = 0$ selbst und diese Ebenen stellen die einzigen speziellen Komplexe der Schaar dar. Die vierte Ebene wird gegeben durch die Verbindungsebene der drei Schnittpunkte $L_i L_k$, hat also die Gleichung

$$(\pi' S'_{13} S'_{24} S'_{14} S'_{23})(\pi' S'_{14} S'_{23} S'_{12} S'_{34})(\pi' S'_{12} S'_{34} S'_{13} S'_{24}) = 0.$$

Nach einiger Rechnung führt dies auf

$$E = 14 - 41 + 32 - 23 = 0, \quad \ldots\ldots \quad (24)$$

wobei die Bedeutung von rs aus (15) zu entnehmen ist. Man rechnet weiters leicht nach, dass diese Ebene E weder die Geraden 1 bis 4 noch die hierzu assoziierte schneidet.

Wenn wir von Komplex K nur verlangen, dass er durch die vier Geraden 1 bis 4 geht, so gelten für die λ_{rs} nur die ersten drei Gleichungen (20). Man kann dann z.B. λ_{12}, λ_{23} und λ_{34} durch die übrigen λ_{rs} ausdrücken und erhält für K die Gleichung

$$\left.\begin{aligned} \lambda_{13}(12-13) + \lambda_{14}(14-12) + \lambda_{24}(24-23) + \lambda_{21}(21-23) + {} \\ + \lambda_{31}(31-34) + \lambda_{32}(32-34) = 0. \end{aligned}\right\} \quad (25)$$

Hier ist z.B.

$$12-13 = K_{12}\frac{A_1}{A_2} - K_{13}\frac{A_1}{A_3} = \frac{A_1}{A_2} A_{4,23,2\pi} - \frac{A_1}{A_3} A_{2,34,3\pi} = 0 \ . \quad (26)$$

ein Komplex durch alle vier Geraden und nach (25) gibt es also — wie es sein muss — ∞^5 solche lineare Komplexe.

Es ergibt sich von hier aus auch eine Lösung für die Aufgabe: Ermittlung der Gleichung eines linearen Strahlenkomplexes K, der durch neun Geraden

$$a_{ik}, \alpha_{ik}, \ldots, \sigma_{ik}$$

gegeben ist. In Gestalt einer zehnreihigen Determinante

$$D = |a_{12}\, \alpha_{13} \ldots \sigma_{34}\, \pi_{45}| = 0 \quad \ldots\ldots \quad (27)$$

ist natürlich diese Gleichung sofort hin zu schreiben. Wie kann aber K durch die quinären Invarianten dargestellt werden? Hier bekommen wir, wenn wir in (25) für die π_{ik} die fünfte bis neunte Gerade einsetzen, für die sechs homogenen λ_{rs} fünf Gleichungen, können diese λ_{rs} also eliminieren und erhalten eine sechsreihige Determinante für K:

$$K = \begin{vmatrix} (12-13) & (14-12) & \dots\dots & (32-34) \\ (12-13)_5 & (14-12)_5 & \dots\dots & (32-34)_5 \\ \cdot & \cdot & \dots\dots & \cdot \\ \cdot & \cdot & \dots\dots & \cdot \\ (12-13)_9 & (14-12)_9 & \dots\dots & (32-34)_9 \end{vmatrix} = 0 \quad . \quad (28)$$

Hier bedeutet z.B. $(12-13)_5$, dass für π_{ik} in (26) die Koordinaten der fünften Geraden eingesetzt sind. Wir ersehen hieraus, dass

$$(A_1\,A_2\,A_3\,A_4)^6 \,.\, K$$

ein Polynom von Komitanten des Typus (1) wird.

Mathematics. — *Ueber assoziierte Geraden bei Regelflächen im R_4* Von R. WEITZENBÖCK.

(Communicated at the meeting of February 24, 1940.)

Uebersicht. Zu vier erzeugenden Geraden G_1 bis G_4 einer zweidimensionalen Regelfläche F im vierdimensionalen Raume R_4 lässt sich die „assoziierte" Gerade G_5 konstruieren. Wenn die vier G_i gegen eine Erzeugende a von F konvergieren, nähert sich auch die assoziierte Gerade G_5 einer Grenzlage g. Man erhält so — von Ausnahmen abgesehen — zu jeder Erzeugenden a von F eine assoziierte Gerade g; sie bilden in ihrer Gesamtheit die zu F „assoziierte Regelfläche".

Ich ermittle hier die Gleichung von g unter der Voraussetzung, dass die homogenen Linienkoordinaten a_{ik} der Erzeugenden a von F als reguläre Funktionen eines Parameters gegeben sind. Die hierzu erforderlichen Rechnungen sind nicht ganz mühelos, man hat bis zu den Gliedern dreissigster Ordnung in den benutzten Potenzreihen vorzudringen.

§ 1.

Die Geraden einer einparametrigen Schaar bildem im R_4 die Erzeugenden einer zweidimensionalen Regelfläche F. Wir denken uns F analytisch gegeben durch die zehn homogenen Koordinaten a_{ik} einer Erzeugenden, wobei diese a_{ik} stetige und genügend oft differenzierbare Funktionen eines komplexen Parameters t sind:

$$a_{ik} = a_{ik}(t) \qquad (i, k = 1, 2, \ldots, 5).$$

Da die a_{ik} Linienkoordinaten sind, verschwinden die fünf quadratischen Ausdrücke

$$\Sigma a_{23} a_{45}, \ \Sigma a_{13} a_{45}, \ldots, \Sigma a_{12} a_{34}$$

für alle Werte von t. Wenn b mit a äquivalent ist, lässt sich dies symbolisch ausdrücken durch das Nullsein der Kovariante

$$4 \,.\, \Sigma x_1 a_{23} a_{45} = (xa^2 b^2) = M'_{00} \equiv 0 \qquad \{x, t\}.$$

Hieraus ergeben sich durch fortgesetztes Differenzieren nach t eine Reihe von Gleichungen, die für das Folgende von grundlegender Bedeutung sind.

Wir setzen

$$(a_1)_{ik} = \frac{d\, a_{ik}(t)}{dt}, \ (a_2)_{ik} = \frac{d^2 a_{ik}(t)}{dt^2}, \ldots$$

und deuten $a, a_1, a_2, \ldots$ an durch die Ziffern $0, 1, 2, \ldots$. Dann ergibt sich, identisch in t:

$$\left.\begin{array}{r}
M'_{00} = 0 \\
M'_{01} = 0 \\
M'_{11} + M'_{02} = 0 \\
3\,M'_{12} + M'_{03} = 0 \\
3\,M'_{22} + 4\,M'_{13} + M'_{04} = 0 \\
10\,M'_{23} + 5\,M'_{14} + M'_{05} = 0 \\
10\,M'_{33} + 15\,M'_{24} + 6\,M'_{15} + M'_{06} = 0 \\
35\,M'_{34} + 21\,M'_{25} + 7\,M'_{16} + M'_{07} = 0 \\
35\,M'_{44} + 56\,M'_{35} + 28\,M'_{26} + 8\,M'_{17} + M'_{08} = 0 \\
126\,M'_{45} + 84\,M'_{36} + 36\,M'_{27} + 9\,M'_{18} + M'_{09} = 0 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}\right\} \quad \ldots \quad (1)$$

Hier sind die $(a_1)_{ik}, (a_2)_{ik}, \ldots$ die Koordinaten von linearen Ebenenkomplexen $K_1, K_2 \ldots$. Der Komplex K_h ist speziell, stellt also eine Gerade dar, wenn $M'_{hh} = 0$ gilt. So ist z.B. der Komplex K_1 speziell für $M'_{11} = -M'_{02} = 0$; wegen $M'_{01} = 0$ enthält K_1 jede Ebene durch die Erzeugende a_{ik}.

Wenn die Fläche F eine Torse ist, d.h. ihre Erzeugenden die Tangenten einer Raumkurve

$$y_i = y_i(t) \qquad (i = 1, 2, \ldots, 5)$$

sind, so findet man

$$\begin{aligned}
a_{ik} &= (y\,y')_{ik} \\
(a_1)_{ik} &= (y\,y'')_{ik} \\
(a_2)_{ik} &= (y'\,y'')_{ik} + (y\,y''')_{ik} \\
(a_3)_{ik} &= 2\,(y'\,y''')_{ik} + (y\,y^{IV})_{ik} \\
&\text{u. s. f.}
\end{aligned}$$

woraus sich ausser $M'_{00} = 0$, $M'_{01} = 0$ noch weiters ergibt:

$$M'_{11} = 0, \; M'_{02} = 0, \; M'_{12} = 0, \; M'_{03} = 0.$$

Dagegen sind die M'_{ik} mit $i + k \geqq 4$ im Allgemeinen nicht Null.

Liegt F gänzlich in einem linearen R_3, so ist M'_{02} bis auf einen von t abhängigen Faktor eine Linearform in x mit konstanten Koeffizienten:

$$M'_{02} = (v'\,x) \cdot f(t) \quad \ldots\ldots\ldots \quad (2)$$

Wir wollen im Folgenden diese beiden Fälle ausschliessen, *also voraussetzen, dass die Fläche F weder abwickelbar sei noch zur Gänze in einem linearen R_3 liege.*

§ 2.

Vier Geraden 1 bis 4 mit den Koordinaten a_{ik}, α_{ik}, p_{ik} und m_{ik} haben im R_4 die vier Invarianten

$$\left.\begin{aligned} A_1 &= (ma^2\,p^2)\,(ma^2\,\alpha^2) = (mS'_{13})\,(mS'_{12}) \\ A_2 &= (a\alpha^2\,p^2)\,(a\alpha^2\,m^2) = (aS'_{23})\,(aS'_{24}) \\ A_3 &= (\alpha a^2\,p^2)\,(\alpha p^2\,m^2) = (\alpha S'_{13})\,(\alpha S'_{34}) \\ A_4 &= (pa^2\,m^2)\,(p\alpha^2\,m^2) = (pS'_{14})\,(pS'_{24}) \end{aligned}\right\} \quad \ldots\ldots \quad (3)$$

und die Gleichung der zu 1 bis 4 assoziierten Geraden G_5 wird gegeben durch [1]):

$$G_5 = G_1\,A_2\,A_3\,A_4 + G_2\,A_1\,A_3\,A_4 + G_3\,A_1\,A_2\,A_4 + G_4\,A_1\,A_2\,A_3 = 0. \quad (4)$$

Wir nehmen jetzt als Geraden 1 bis 4 vier Erzeugende von F, entsprechend den vier Parameterwerten α, β, γ und δ. Es ist dann also

$$\left.\begin{aligned} G_1 &= a_{ik} + \frac{\alpha}{1!}(a_1)_{ik} + \frac{\alpha^2}{2!}(a_2)_{ik} + \ldots = 0_{ik} + \frac{\alpha}{1!}1_{ik} + \frac{\alpha^2}{2!}2_{ik} + \ldots \\ G_2 &= a_{ik} + \frac{\beta}{1!}(a_1)_{ik} + \frac{\beta^2}{2!}(a_2)_{ik} + \ldots = 0_{ik} + \frac{\beta}{1!}1_{ik} + \frac{\beta^2}{2!}2_{ik} + \ldots \\ &\text{u. s. f. mit } \gamma \text{ und } \delta. \end{aligned}\right\} \quad . \quad (5)$$

Für diese G_i wollen wir (4) berechnen. Hierzu haben wir vorerst die A_i nötig.

Wir berechnen A_1. Aus (5) finden wir zuerst für S'_{12}, wenn wir aus allen Gliedern den Faktor $\frac{1}{2}(\alpha-\beta)^2$ absondern:

$$\left.\begin{aligned} S'_{12} = M'_{02} &+ \left[\frac{1}{3}(\alpha+\beta)\,M'_{03}\right] + \left[\frac{1}{12}(\alpha+\beta)^2\,M'_{04} + \frac{1}{3}\,\alpha\beta\,M'_{13}\right] + \\ &+ \left[\frac{1}{60}(\alpha+\beta)(\alpha^2+\alpha\beta+\beta^2)\,M'_{05} + \frac{1}{12}\,\alpha\beta\,(\alpha+\beta)\,M'_{14}\right] + \\ &+ \left[\frac{1}{360}(\alpha^2+\alpha\beta+\beta^2)^2\,M'_{06} + \frac{1}{30}\,\alpha\beta\,(\alpha+\beta)^2\,M'_{15} + \frac{1}{24}\,\alpha^2\beta^2\,M'_{24}\right] + \\ &+ \Big[\frac{2}{7!}(\alpha+\beta)(\alpha^2+\beta^2)(\alpha^2+\alpha\beta+\beta^2)\,M'_{07} + \\ &\qquad + \frac{1}{360}\,\alpha\beta\,(\alpha+\beta)(\alpha^2+\alpha\beta+\beta^2)\,M'_{16} + \frac{1}{120}\,\alpha^2\beta^2\,(\alpha+\beta)\,M'_{25}\Big] + \\ &+ \Big[\frac{2}{8!}(\alpha+\beta)^2(\alpha^2+\beta^2)^2\,M'_{08} + \frac{2}{7!}\,\alpha\beta\,(\alpha^2+\alpha\beta+\beta^2)^2\,M'_{17} + \\ &\qquad + \frac{1}{720}\,\alpha^2\beta^2\,(\alpha+\beta)^2\,M'_{26} + \frac{1}{360}\,\alpha^3\beta^3\,M'_{35}\Big] + \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad (6)$$

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1) Vgl. diese Proceedings **42**, 248 (1939).

Wir haben hier die Glieder gleich hohen Grades in α, β zwischen eckige Klammern gesetzt. Aus (6) lesen wir ab, dass

$$(x M'_{02}) = (x a^2 a_2^2) = (x 0^2 2^2) = 0$$

die Gleichung des längs der Erzeugenken a_{ik} berührenden R_3 darstellt.

Vertauschen wir in (6) β mit γ, so ergibt sich S'_{13} und damit lässt sich nach der ersten Gleichung der Formeln (3) A_1 berechnen. Wenn wir zunächst m_{ik} allgemein lassen, so ergibt sich nach Abspaltung des Faktors $\frac{1}{3}(\beta-\gamma)$, also bis auf den Faktor

$$\frac{1}{12}(\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma) \quad \ldots\ldots\ldots \quad (7)$$

für $(m S'_{13})(m S'_{12})$ die Reihe

$$\left.\begin{aligned}
(m S'_{13})(m S'_{12}) = m_{02,03} &+ \left[\frac{1}{4}(2\alpha+\beta+\gamma)\, m_{02,04} + \alpha\, m_{02,13}\right] + \\
&+ \left[\frac{1}{12}(\alpha+\beta)(\alpha+\gamma)\, m_{03,04} + \frac{1}{3}\alpha^2 m_{03,13} + \right.\\
&+ \frac{1}{4}\{\alpha^2+\alpha(\beta+\gamma)\}\, m_{02,14} + \\
&+ \left.\frac{1}{20}\{2\alpha^2+2\alpha(\beta+\gamma)+\beta^2+\beta\gamma+\gamma^2\}\, m_{02,05}\right] + \\
&+ \left[\frac{1}{120}\{2\alpha^2+\alpha(\beta+\gamma)+\beta^2+\gamma^2\}(\alpha+\beta+\gamma)\, m_{02,06} + \right.\\
&+ \frac{1}{10}\{\alpha^2+2\alpha(\beta+\gamma)+\beta^2+\beta\gamma+\gamma^2\}\, m_{02,15} + \\
&+ \frac{1}{8}\alpha^2(\beta+\gamma)\, m_{02,24} + \frac{1}{60}(\alpha+\beta)(\alpha+\gamma)(\alpha+\beta+\gamma)\, m_{03,05} + \\
&+ \left.\frac{1}{12}\alpha(\alpha+\beta)(\alpha+\gamma)\, m_{03,14} + \frac{1}{12}\alpha(\alpha^2-\beta\gamma)\, m_{04,13}\right] + \\
&+ \left[\varrho_0\, m_{02,07} + \varrho_1\, m_{02,16} + \varrho_2\, m_{02,25} + \frac{1}{12}\alpha^2\beta\gamma\, m_{13,14}\right] + \ldots\ldots
\end{aligned}\right\} \quad (8)$$

In (8) haben wir schliesslich nach (5) für m_{ik} die Reihe

$$0_{ik} + \frac{\delta}{1!}1_{ik} + \frac{\delta^2}{2!}2_{ik} + \ldots$$

einzusetzen um die verlangte Potenzreihe für A_1 zu erhalten. Die Rechnung ergibt, wenn wir für das Differenzenprodukt

$$\Pi = (\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta) \quad \ldots\ldots \quad (9)$$

die Abkürzung Π gebrauchen und

$$Q = 0_{13,23} = (aa_1^2\, a_3^2)\,(aa_2^2\, a_3^2) \quad \ldots\ldots \quad (10)$$

setzen:

$$A_1 = -\tfrac{1}{72}\, Q\,.\,\Pi\,.\,(\alpha-\beta)\,(\alpha-\gamma)\,(\alpha-\delta) + \ldots.$$

Hier bedeuten die Punkte Glieder vom zehnten und höheren Grade in den $\alpha, \beta, \gamma, \delta$.

Wenn wir z.B. α mit β vertauschen, werden nach (5) G_1 und G_2 vertauscht, A_1 geht dann in $-A_2$, A_2 in $-A_1$, A_3 in $-A_3$ und A_4 in $-A_4$ über. Auf diese Weise ergeben sich für die A_i aus der zuletzt angeschriebenen Gleichung die Reihen:

$$\left.\begin{aligned} A_1 &= -\tfrac{1}{72}\, Q\,.\,\Pi\,.\,(\alpha-\beta)\,(\alpha-\gamma)\,(\alpha-\delta) + \ldots. \\ A_2 &= +\tfrac{1}{72}\, Q\,.\,\Pi\,.\,(\alpha-\beta)\,(\beta-\gamma)\,(\beta-\delta) + \ldots. \\ A_3 &= -\tfrac{1}{72}\, Q\,.\,\Pi\,.\,(\alpha-\gamma)\,(\beta-\gamma)\,(\gamma-\delta) + \ldots. \\ A_4 &= +\tfrac{1}{72}\, Q\,.\,\Pi\,.\,(\alpha-\delta)\,(\beta-\delta)\,(\gamma-\delta) + \ldots. \end{aligned}\right\} \quad \ldots \quad (11)$$

§ 3.

Bevor wir nach Gleichung (4) G_5 berechnen, sind einige Bemerkungen bezüglich der Grösse Q von (10) zu machen.

Sind a_{ik}, α_{ik}, p_{ik}, m_{ik} und π_{ik} die Koordinaten von fünf linearen Komplexen, so ist

$$J = a_{\alpha p, m\pi} = (a\alpha^2\, p^2)\,(am^2\, \pi^2) = 16\,.\,\Sigma\, a_{ik}\, \alpha_{i_2 i_3}\, p_{i_4 i_5}\, m_{k_2 k_3}\, \pi_{k_4 k_5}$$

eine projektive Komitante dieser Komplexe. Sie ist symmetrisch in α und p und in m und π, schief-symmetrisch in den Paaren αp und $m\pi$:

$$J = a_{\alpha p, m\pi} = -\, a_{m\pi, \alpha p}.$$

Schliesslich ist J zyklisch-symmetrisch in a, α, p und ebenso in a, m, π:

$$a_{\alpha p, m\pi} + \alpha_{pa, m\pi} + p_{a\alpha, m\pi} = 0$$

und ebenso mit a, m, π.

Diese Symmetrieeigenschaften im Vereine mit den Gleichungen (1) ergeben eine Reihe von Reduktionen bei denjenigen Ausdrücken, die aus J entstehen, wenn wir an Stelle der

$$a_{ik}, \alpha_{ik}, p_{ik}, \ldots. \text{ die Reihen } a_{ik} = 0_{ik}, (a_1)_{ik} = 1_{ik}, (a_2)_{ik} = 2_{ik}, \ldots.$$

nehmen. Es sei dies bei der durch die Gleichung (10) eingeführten Grösse Q näher ausgeführt.

Wenn wir in (8)

$$m_{ik}=0_{ik}+\frac{\delta}{1!}1_{ik}+\frac{\delta^2}{2!}2_{ik}+\dots.$$

einsetzen, so werden die Glieder vierten Grades in $\alpha, \beta, \gamma, \delta$, da

$$0_{0i,rs}=0$$

für alle i, r und s gilt (die 0_{ik} sind Linienkoordinaten!), gegeben durch

$$\left.\begin{aligned}O_4=&\frac{1}{12}\alpha^2\beta\gamma\,0_{13,14}+\\&+\frac{\delta}{1!}\Big[\sigma_1 1_{02,06}+\sigma_2 1_{02,15}+\sigma_3 1_{02,24}+\sigma_4 1_{03,05}+\\&+\frac{1}{12}\alpha(\alpha+\beta)(\alpha+\gamma)1_{03,04}+\frac{1}{12}\alpha(\alpha^2-\beta\gamma)1_{04,13}\Big]+\\&+\frac{\delta^2}{2!}\Big[\frac{1}{12}(\alpha+\beta)(\alpha+\gamma)2_{03,04}+\frac{1}{3}\alpha^2 2_{03,13}+\\&+\frac{1}{4}(\alpha^2+\alpha(\beta+\gamma))2_{02,14}+\sigma_5 2_{02,05}\Big]+\\&+\frac{\delta^3}{3!}\Big[\frac{1}{4}(2\alpha+\beta+\gamma)3_{02,04}+\alpha 3_{02,13}\Big]+\\&+\frac{\delta^4}{4!}\cdot 4_{02,03}.\end{aligned}\right\}\quad (12)$$

Die hier auftretenden Koeffizienten K:

$$i_{jk,mn}\qquad (i+j+k+m+n=9)$$

sind alle auf das Q von Gleichung (10) reduzierbar oder verschwinden.

Sei z.B. $K=2_{02,05}$. Die zyklische Symmetrie bezüglich 2, 0, 5 gibt hier

$$2_{02,05}=-0_{02,25}-5_{02,02}=0,$$

denn $0_{02,05}$ verschwindet, da 0_{ik} Linienkoordinaten sind und $5_{02,02}$ ist Null wegen der schiefen Symmetrie in den Paaren 02, 02.

Wenn $K=1_{02,15}$ ist, so haben wir zufolge der dritten der Gleichungen (1)

$$K=1_{02,15}=-1_{11,15}$$

und dies ist Null, da allgemein $r_{rr,ik}=0$ gilt, wie aus der zyklischen Symmetrie in r, r, r sofort folgt.

Bei $K=4_{02,03}$ gibt die zyklische Symmetrie bezüglich 4, 0, 2:

$$K=4_{02,03}=-0_{24,03}-2_{04,03}=-2_{04,03}.$$

Aus (1) haben wir

$$04 = -3.22 - 4.13,$$

also wird

$$4_{02,03} = 3.2_{22,03} + 4.2_{13,03} = 4.2_{13,03}.$$

Hier ist, wieder wegen der zyklischen Symmetrie bezüglich 2, 1, 3:

$$2_{13,03} = -1_{23,03} - 3_{12,03}.$$

Der letzte Term wird nach (1) wegen $03 = -3.12$ gleich $3.3_{12,12}$, also Null. Beim ersten Term der rechten Seite gibt die zyklische Symmetrie bezüglich 1, 0, 3:

$$1_{23,03} = -0_{23,13} - 3_{23,01} = 0_{13,23} = Q,$$

da $3_{23,01}$ wegen 01 nach (1) wegfällt. Also wird schliesslich

$$2_{13,03} = -Q \text{ und } 4_{02,03} = -4Q.$$

Auf diese Weise sind alle Koeffizienten in (12) zu reduzieren. Man erhält:

$$\left.\begin{array}{l} 0_{13,14} = -2Q \\ 1_{02,06} = 0,\ 1_{02,15} = 0,\ 1_{02,04} = 0,\ 1_{03,05} = 0,\ 1_{03,14} = 2Q,\ 1_{04,13} = -2Q \\ 2_{03,04} = -4Q,\ 2_{03,13} = Q,\ 2_{02,14} = -\frac{4}{3}Q,\ 2_{02,05} = 0 \\ 3_{02,04} = 4Q,\ 3_{02,13} = 0 \\ 4_{02,03} = -4Q. \end{array}\right\} \quad (13)$$

Setzt man dies in (12) ein, so wird

$$O_4 = -\tfrac{1}{6}\, Q\, (\alpha-\delta)^2 (\beta-\delta)(\gamma-\delta) \quad . \quad . \quad . \quad . \quad . \quad (14)$$

was schliesslich mit (7) zusammen zu den Gleichungen (11) führt. Diese letzteren ergeben dann

$$\left.\begin{array}{l} A_2 A_3 A_4 = -\dfrac{1}{72^3}\, Q^3 \Pi^4 (\beta-\gamma)(\beta-\delta)(\gamma-\delta) + \dots \\[2ex] A_1 A_3 A_4 = +\dfrac{1}{72^3}\, Q^3 \Pi^4 (\alpha-\gamma)(\alpha-\delta)(\gamma-\delta) + \dots \\[2ex] A_1 A_2 A_4 = -\dfrac{1}{72^3}\, Q^3 \Pi^4 (\alpha-\beta)(\alpha-\delta)(\beta-\delta) + \dots \\[2ex] A_1 A_2 A_3 = +\dfrac{1}{72^3}\, Q^3 \Pi^4 (\alpha-\beta)(\alpha-\gamma)(\beta-\gamma) + \dots \end{array}\right\} \quad . \quad . \quad (15)$$

wo die Punkte Glieder andeuten, die wenigstens vom 28. Grade in $\alpha, \beta, \gamma, \delta$ sind.

Man zeigt leicht an einem Beispiel, dass die Invariante $Q = 0_{13,23}$ im allgemeinen von Null verschieden ist.

Schliesslich setzen wir (15) in (4) ein, wobei die G_i durch die Reihen (5) gegeben sind. Als Glieder niedrigster Ordnung ergeben sich in den Reihen, mit denen

$$m_{ik} = 0_{ik}, 1_{ik}, 2_{ik}, \ldots .$$

zu multiplizieren sind, bis auf den Faktor $-\frac{1}{72^3} Q^3 \Pi^4$ die Ausdrücke

$$\left.\begin{aligned} S_m = \alpha^m (\beta-\gamma)(\beta-\delta)(\gamma-\delta) - \beta^m (\alpha-\gamma)(\alpha-\delta)(\gamma-\delta) + \\ + \gamma^m (\alpha-\beta)(\alpha-\delta)(\beta-\delta) - \delta^m (\alpha-\beta)(\alpha-\gamma)(\beta-\gamma). \end{aligned}\right\} \quad . \quad . \quad . \quad (16)$$

Man rechnet leicht nach, dass S_0, S_1 und S_2 identisch wegfallen, dass dagegen $S_3 \not\equiv 0$ wird, sodass die Reihenentwicklung für G_5 mit den Gliedern dreissigster Ordnung beginnt. Hieraus schliesst man, dass bei $Q \neq 0$

$$lim\, G_5 (\alpha \to 0, \beta \to 0, \gamma \to 0, \delta \to 0) = g_{ik}$$

eine Linearkombination von 0_{ik}, 1_{ik}, 2_{ik} und 3_{ik} sein wird, wobei 3_{ik} mit dem Koeffizienten Q^3 versehen ist.

§ 4.

Man wird so zu dem Ansatz

$$g_{ik} = 0_{ik} + \lambda \,.\, 1_{ik} + \mu \,.\, 2_{ik} + \nu \,.\, 3_{ik} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

geführt. Die Bedingung dafür, dass die g_{ik} Linienkoordinaten sind, lassen sich dann nach (1) zusammenfassen zu

$$\begin{aligned} M'_{11} \lambda^2 + M'_{22} \mu^2 + M'_{33} \nu^2 + 2\mu M'_{02} + 2\nu M'_{03} + \\ + 2\lambda\mu M'_{12} + 2\lambda\nu M'_{13} + 2\mu\nu M'_{23} = 0. \end{aligned}$$

Drückt man hier M'_{11}, M'_{22} und M'_{12} nach (1) durch die übrigen M'_{ik} aus, so bekommt man

$$\left.\begin{aligned} M'_{02}(2\mu-\lambda^2) + M'_{03}(2\nu - \tfrac{2}{3}\lambda\mu) - \tfrac{1}{3}\mu^2 M'_{04} + M'_{13}(2\lambda\nu - \tfrac{4}{3}\mu^2) + \\ + 2\mu\nu M'_{23} + \nu^2 M'_{33} = 0. \end{aligned}\right\} \quad . \quad . \quad (18)$$

Dies ist eine lineare Beziehung zwischen den sechs Räumen R_3, deren Koordinaten die M'_{ik} sind. Nennen wir in der Matrix

$$\| M'_{02}\, M'_{03}\, M'_{04}\, M'_{13}\, M'_{23}\, M'_{33} \|$$

die fünfreihigen Determinaten Θ_{ik}, so gilt

$$\Sigma M'_{ik} \Theta_{ik} = \Theta_{02} M'_{02} + \Theta_{03} M'_{03} + \ldots + \Theta_{33} M'_{33} = 0 \quad . \quad . \quad (19)$$

und dies muss bis auf einen Proportionalitätsfaktor mit (18) identisch

sein. Es gilt also vorerst die Ausdrücke Θ_{ik} zu berechnen. Wir setzen (Vgl. (10)):

$$Q = 0_{13,23}, \quad R = 0_{13,33}, \quad S = 0_{23,33}, \quad T = 2_{04,33}. \quad . \quad . \quad . \quad (20)$$

Dann ist z.B.

$$\Theta_{23} = \sum_i (0^2\,2^2)_i\,((03)\,(04)\,(13)\,(33))_i = \begin{vmatrix} 0_{03} & 0_{04} & 0_{13} & 0_{33} \\ 0_{03} & 0_{04} & 0_{13} & 0_{33} \\ 2_{03} & 2_{04} & 2_{13} & 2_{33} \\ 2_{03} & 2_{04} & 2_{13} & 2_{33} \end{vmatrix} =$$

$$= 4.0_{13,33}\,2_{03,04} = -16\,R\,Q.$$

Auf diese Weise bekommt man nach einiger Rechnung:

$$\left.\begin{array}{ll} \Theta_{02} = 4\,(\tfrac{1}{2}\,R\,T - S^2) & \Theta_{13} = 16\,Q\,S \\ \Theta_{03} = -4\,(Q\,T + \tfrac{2}{3}\,R\,S) & \Theta_{23} = 16\,Q\,R \\ \Theta_{04} = -\tfrac{4}{3}\,R^2 & \Theta_{33} = 16\,Q^2 \end{array}\right\} \quad . \quad . \quad . \quad (21)$$

Wählen wir also als Proportionalitatsfaktor $\frac{1}{4}\,\varrho^2$, so liefert die Gegenüberstellung von (18) und (19) die sechs Gleichungen

$$\left.\begin{array}{r} 2\,\mu - \lambda^2 = \varrho^2\,(\tfrac{1}{2}\,R\,T - S^2) \\ -2\,\nu + \tfrac{2}{3}\,\lambda\,\mu = \varrho^2\,(Q\,T + \tfrac{2}{3}\,R\,S) \\ \mu^2 = \varrho^2\,R^2 \\ 2\,\lambda\,\nu - \tfrac{4}{3}\,\mu^2 = 4\,\varrho^2\,Q\,S \\ \mu\,\nu = -2\,\varrho^2\,Q\,R \\ \nu^2 = 4\,\varrho^2\,Q^2 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Hieraus folgen

$$\mu = \mp\,\varrho\,R \quad , \quad \nu = \pm\,2\,\varrho\,Q$$

womit auch die vorletzte Gleichung (22) befriedigt ist. Aus $(22)_2$ und $(22)_4$ kann man dann ϱ und λ berechnen:

$$\varrho = \mp\,\frac{36\,Q^2}{2\,R^3 + 12\,Q\,R\,S + 9\,Q^2\,T} \quad , \quad \lambda = -\frac{12\,Q\,(3\,Q\,S + R^2)}{2\,R^3 + 12\,Q\,R\,S + 9\,Q^2\,T}.$$

Mit diesen Werten wird auch $(22)_1$ befriedigt, sodass sich schliesslich unter der Voraussetzung

$$P = 2\,R^3 + 12\,Q\,R\,S + 9\,Q^2\,T \neq 0. \quad . \quad . \quad . \quad . \quad (23)$$

für die gesuchte Gerade g ergibt:

$$\left.\begin{array}{l} g_{ik} = a_{ik}\,(2R^3 + 12\,Q\,R\,S + 9\,Q^2\,T) - (a_1)_{ik}\,.\,12\,Q\,(3\,Q\,S + R^2) + \\ \qquad\qquad + (a_2)_{ik}\,.\,36\,Q^2\,R - (a_3)_{ik}\,.\,72\,Q^3 \end{array}\right\}. \quad (24)$$

Diese Gerade g_{ik} ist die „assoziierte" der Erzeugenden a_{ik}.

Chemistry. — *Der Einfluss des Dispersitätsgrades auf physikalisch-chemische Konstanten.* (Neunte Mitteilung.) Von ERNST COHEN und J. J. A. BLEKKINGH Jr.

(Communicated at the meeting of February 24, 1940.)

Polarographische Analyse der bei 25.00° C. gesättigten Lösung des Bariumsulfats.

Einleitung.

In unserer siebenten [1]) und achten [2]) Mitteilung wurde der Nachweis geführt, dass der Löslichkeit des Bariumsulfats in Wasser bei bestimmter Temperatur und bestimmtem Drucke (1 Atm.) ein bestimmter Wert zukommt, welcher nicht abhängt von dem Dispersitätsgrade des Bodenkörpers. Dieser Beweis wurde erbracht durch Messung des elektrischen Leitvermögens der gesättigten Lösung des chemisch und physikalisch reinen Sulfats. In der vorliegenden Abhandlung beschreiben wir unsere Versuche zur Erhärtung dieses Ergebnisses auf polarographischem Wege. Zu diesem Zweck verwandten wir das polarographische Analysenverfahren, welches uns auch in stand setzte, die Ba-Ionenkonzentration der gesättigten Lösung quantitativ zu ermitteln, und aus dieser die Löslichkeit des Bariumsulfats in Wasser zu berechnen. Sämtliche Versuche wurden bei 25.00° C. (und 1 Atm. Druck) ausgeführt.

Ueber die Bariumbestimmung auf polarographischem Wege haben J. HEYROVSKÝ und Mitarbeiter [3]) bereits einige Untersuchungen angestellt. Indes entsprechen die Schlüsse, welche sie daraus betreffs der Löslichkeit des Bariumsulfats und des „Alterns" des Niederschlags ziehen, dem Tatbestande keineswegs, da erstens das von ihnen verwendete Bariumsulfat nicht chemisch rein, zweitens aber ihre Lösungen nicht vorher filtriert waren.

A. Die Apparatur.

1. Eine ausführliche Beschreibung des auch von uns verwendeten Verfahrens gibt J. HEYROVSKÝ in dem Kapitel „Polarographie" des Werkes „Physikalische Methoden der analytischen Chemie" von W. BÖTTGER,

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1) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, 32 (1940).
2) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, 189 (1940).
3) Collection **1**, 36 (1929); Phil. Mag. (6) **45**, 303 (1923).

Leipzig 1936, S. 260—323 [1]). Für nähere Angaben verweisen wir auf die Dissertation von J. J. A. BLEKKINGH Jr., Utrecht 1938.

Der von uns verwendete Apparat von HEYROVSKÝ war von der Firma V. und J. NEJEDLY in Prag verfertigt. Das benutzte Spulengalvanometer war nach JULIUS [2]) erschütterungsfrei aufgehängt. Es liessen sich bei maximaler Empfindlichkeit Stromstärken bis 10^{-9} Amp. bestimmen. Die weiter unten abgebildeten Polarogramme sind in ihrer ursprünglichen Grösse reproduziert worden. (Empfindlichkeit des Galvanometers $^2/_3$). Unsere Tropfelektrode bestand aus einer mit doppelt destilliertem Quecksilber gefüllten Kapillare aus Jenaer Glas, welche zu einer feinen Spitze ausgezogen war. Wir verwandten bei sämtlichen Versuchen dieselbe Kapillare, während das Quecksilbergefäss des Apparates sich stets 64 cm über dem unteren Ende derselben befand. In dieser Weise war es möglich reproduzierbare Ergebnisse zu erzielen; dies ergibt sich aus unseren Figuren, welche sich auf Versuche beziehen, in denen stets zweimal dieselbe Lösung polarographiert wurde. Das Quecksilber aus dem Behälter strömte langsam durch einen gründlich gereinigten, 1 m langen, dickwandigen Schlauch aus transparantem Pararubber zur Kapillare.

B. Tetramethylammoniumjodid als indifferenter Elektrolyt.

2. Bei den Bestimmungen entfernten wir den Sauerstoff aus unseren Lösungen indem wir während 1 bis 2 Std., in manchen Fällen auch während längerer Zeit, reinen Wasserstoff durch dieselben leiteten. Es wird nämlich der Sauerstoff eher abgeschieden (bei —0.4 und —1.1 Volt) als das Barium (bei —1.95 Volt), was zu einer erhöhten Stromstärke Anlass geben würde.

Als indifferenten Elektrolyten versuchten wir zunächst (nach J. HEYROVSKÝ) Lithiumchlorid bezw. Lithiumchlorat (die Abscheidung des Li^+ erfolgt bei —2.3 Volt). Es gelang uns indes nicht hiermit befriedigende Resultate zu erzielen, wohl infolge der in den Lithiumsalzen vorhandenen Verunreinigung. Aber auch ohne Zusatz eines indifferenten Elektrolyten gelang es uns nicht unser Ziel zu erreichen.

Dagegen stellte sich heraus, dass eine Lösung des Tetramethylammoniumjodids (dieselbe zeigt erst bei —2.6 Volt eine ausgesprochene Leitfähigkeit) sich bei unseren Bestimmungen vorzüglich als indifferenter Elektrolyt eignete. Zwar enthielt das Präparat (pro analysi) von KAHLBAUM, auch nach sorgfältigem Umkristallisieren aus L. W. [3]) noch spuren H^+—(?) und NH_4^+ — Ionen, wie sich aus Fig. 1 aus der Abweichung von einer

[1]) Neuere Literatur: „De polarographische methode met de druppelende kwikelektrode ten dienste van het pharmaceutisch onderzoek" von J. MAAS, Dissertation Amsterdam, 1937, sowie auch „Chemische Analyse mit dem Polarographen" von H. HOHN, Berlin 1937.

[2]) Ann. Physik (3) **56**, 151 (1895).

[3]) Vergl. unsere achte Mitteilung, § 1.

geraden Linie ergibt, welche bereits bei —1.6 Volt anfängt, sowie aus der schwachen Welle bei —2.1 Volt.

Wir stellten uns eine 0.01 n Lösung des Jodids in L.W. her, und gaben stets 45 cc der zu analysierenden Lösung zu 5 cc dieser Jodidlösung. Von dieser Mischung brachten wir etwa 20 cc in das W. G. und leiteten sodann während wenigstens 1 Std. Wasserstoff durch diese Lösung.

Bei sämtlichen Versuchen ist die Endkonzentration somit 0.001 n an Tetramethylammoniumjodid. Unsere Fig. 1 stellt das Ergebnis des blinden Versuchs mit dieser Lösung dar.

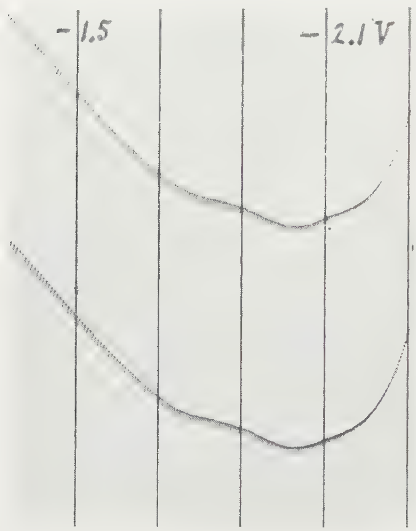

Fig. 1. 0.001 n. Tetramethylammoniumjodidlösung bei 25.00° C.

Fig. 2. Wasser, filtriert durch ein nicht völlig gereinigtes Glasfilter, durch welches vorher eine Bariumsulfatlösung filtriert war.

Ein äusserlich ziemlich sorgfältig gereinigtes Glasfilter gab beim Filtrieren von L. W. durch dasselbe noch Ba^{++}-Ionen an das Wasser ab, welche von vorher auf diesem Filter filtriertem Bariumsulfat herrührten. Dies zeigt sich deutlich in der Fig. 2 an der schwachen Welle bei —1.95 Volt. Wird das Glasfilter aber mit warmer, konzentrierter Schwefelsäure behandelt, und sodann gründlich mit L. W. ausgewaschen, so erhält man ein mit Fig. 1 völlig identisches Polarogramm.

C. *Die Eichung mit Bariumchloridlösungen.*

3. Bevor wir das polarographische Verfahren auf gesättigte Bariumsulfatlösungen anwandten, analysierten wir auf diesem Wege einige Bariumchloridlösungen bekannter Konzentration. Zu diesem Zwecke stellten wir uns Lösungen desselben her, welche 10^{-5} n bzw. 4×10^{-5} n waren. Wir benutzten $BaCl_2 . 2H_2O$ (KAHLBAUM, p. a.), von welchem wir

zunächst festgestellt hatten, dass es genau 2 Mole Kristallwasser enthielt. Zwar enthält dieses Präparat 0.1 % Sr^{++}, sowie Ca^{++}, dies hat indes für unseren Fall keine Bedeutung.

Die Fig. 3 und 4 zeigen die erhaltenen Polarogramme in ihren wahren Dimensionen.

Fig. 3. 10^{-5} n. Bariumchloridlösung.

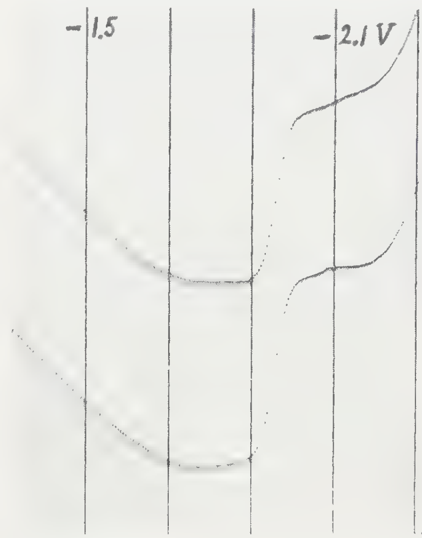

Fig. 4. 4×10^{-5} n. Bariumchloridlösung.

D. Versuche mit filtrierten, gesättigten Bariumsulfatlösungen.

4. Wie bei den Messungen des elektrischen Leitvermögens dieser Lösungen beschrieben wurde (achte Mitteilung), stellten wir uns Suspensionen von nicht zerriebenem, chemisch und physikalisch reinem $BaSO_4$ her, sowie solche des Sulfats, welches im Achatmörser zerrieben war. Im letzteren Falle verwandten wir stets 0.7 g des trocknen Sulfats, welches wir in 150 cc der gesättigten Lösung gaben. Sodann saugten wir die Suspension auf einem Glasfilter ab (achte Mitteilung § 6), und setzten zu 45 cc des Filtrates so schnell als möglich 5 cc der 0.01 n Tetramethylammoniumjodidlösung.

Sollte die Konzentration des Ba^{++} im Filtrat des zerriebenen Sulfats eventuell eine grössere sein, so wird dieselbe durch dieses Verdünnen nicht mehr verschwinden können, da die Lösung ungesättigt geworden ist.

Unsere Fig. 5 und 6 zeigen die Polarogramme der Lösungen, welche in der beschriebenen Art und Weise dargestellt wurden mittels des nicht zerriebenen, bezw. des zerriebenen Bariumsulfats.

Eine Differenz zwischen der Ba^{++}-Ionenkonzentration in diesen und vielen andern Versuchen, war nicht nachweisbar, auch nicht dann, wenn die Lösung während einiger Zeit ruhig gestanden hatte.

5. Aus den Mittelwerten des Grenzstromes (der Höhe der Welle)

und denen bei den Bariumchloridlösungen (Fig. 3 und 4), berechneten wir, dass die bei 25.00° C. gesättigte $BaSO_4$-Lösung 2.08×10^{-5} n ist, somit

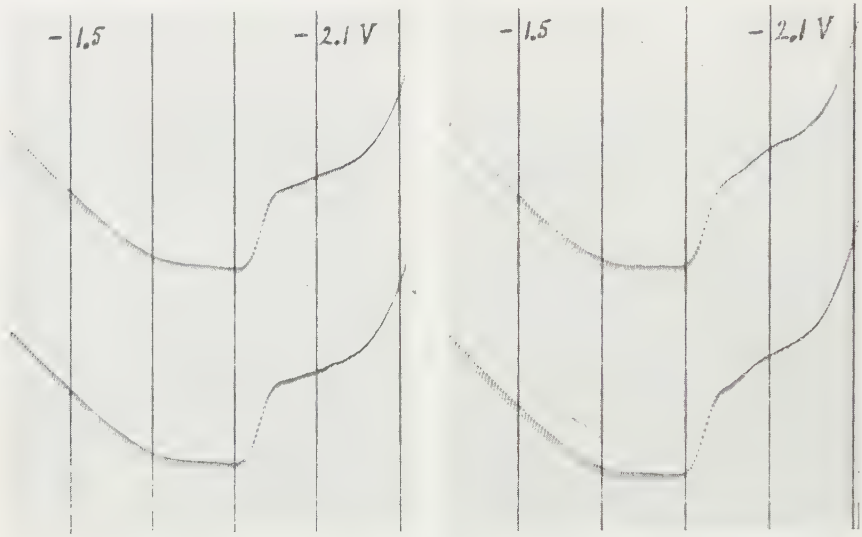

Fig. 5. Nicht-zerriebenes $BaSO_4$ Fig. 6. Zerriebenes $BaSO_4$
Gesättigte $BaSO_4$-Lösungen in Wasser bei 25.00° C.

2.43 mg gelösten $BaSO_4$ pro Liter enthält. Dabei wurde angenommen, dass das sich in Lösung befindliche $BaSO_4$ völlig elektrolytisch dissoziiert ist. Für das Löslichkeitsprodukt (Aktivitätsprodukt) $Ba^{++} \times SO''_4$ bei 25.00° C. ergibt sich hieraus der Wert $\mathbf{1.08} \times 10^{-10}$.

Dieser Wert stimmt überein mit demjenigen, welchen A. C. MELCHER[1]) mittels des elektrischen Leitvermögens gefunden hat.

E. Versuche mit nicht-filtrierten Bariumsulfatsuspensionen.

6. Auch einige Versuche mit nicht-filtrierten Suspensionen des Bariumsulfats, welches zuvor nicht, bezw. wohl zerrieben war, haben wir ausgeführt. Nachdem während 1 Std. Wasserstoff eingeleitet war, nahmen wir die Polarogramme auf, welche in Fig. 7 und 8 reproduziert sind.

Auch hier ergeben sich gleiche Ba^{++}-Ionenkonzentrationen. Die zum Bodenkörper gehörigen Gegenionen werden offenbar nicht bei derselben Potentialdifferenz ausgeschieden, wie die freien Ba^{++}-Ionen der Lösung.

Handelt es sich darum die Konzentration des Sulfats mit der in den früheren Bestimmungen (in welchen 45 cc gesättigter Lösung verwendet wurden) zu vergleichen, so haben wir den 45/50 Teil des Ergebnisses in Rechnung zu ziehen, da jetzt ein Ueberschuss an Bodenkörper vorhanden ist, so dass sich 50 cc einer gesättigten Lösung bilden. Wir finden dann, dass diese Lösung 2.31×10^{-5} n ist, somit **2.71** mg Bariumsulfat pro Liter

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[1]) J. Am. Chem. Soc. **32**, 50 (1910).

enthält. Dabei ist wiederum angenommen worden, dass völlige Dissoziation vorliegt.

Die Löslichkeit des $BaSO_4$ in einer 0.001 n Tetramethylammoniumjodid-

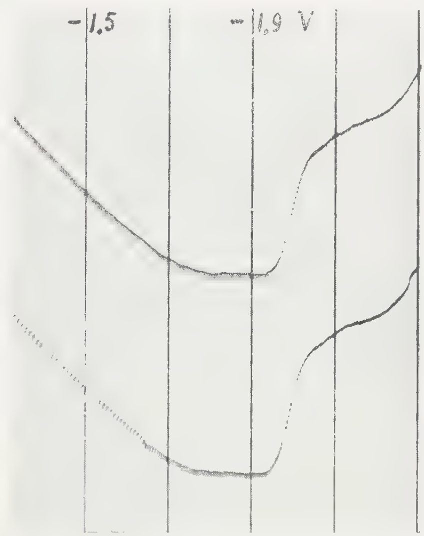

Fig. 7. Nicht-zerriebenes $BaSO_4$ Fig. 8. Zerriebenes $BaSO_4$

Suspensionen von $BaSO_4$ in einer 0.001 n. wässerigen Tetramethylammonium-jodidlösung bei 25.00° C.

lösung ist somit bei 25.00° C. (2.71 — 2.43) = 0.28 mg pro Liter, höher (etwa 10 %), als die in Wasser. Auch hieraus ergibt sich, dass die Schlüsse von J. HEYROVSKÝ [1]) dem Tatbestande nicht entsprechen.

Zusammenfassung.

In unserer 7., 8. und 9. Mitteilung wurde sowohl mittels Leitfähigkeitsmessungen, wie auch auf polarographischem Wege der Nachweis erbracht, dass sich ein Einfluss des Dispersitätsgrades auf die Löslichkeit eines kristallisierten, chemisch und physikalisch reinen Stoffes ($BaSO_4$) nicht feststellen lässt, falls man durch geeignetes Filtrieren der gesättigten Lösung desselben dem Auftreten des „Dispersitätseffektes" vorbeugt.

Mittels polarographischer Analyse erfolgte dann weiter die Feststellung, dass die Löslichkeit des Bariumsulfats bei 25.00° C. 2.43 mg pro Liter beträgt, und dementsprechend das Löslichkeitsprodukt 1.08×10^{-10}.

VAN 'T HOFF-*Laboratorium.*

Utrecht, Februar 1940.

[1]) Collection **1**, 36 (1929); Phil. Mag. (6) **45**, 303 (1923).

Geology. — *Remarks on the geology of Colombia and Venezuela.* I. *The age of the non-fossiliferous slates and of the metamorphic schists.* By L. RUTTEN.

(Communicated at the meeting of February 24, 1940.)

The author has been in the necessity of studying, during the past months, the geologic literature on Colombia and Venezuela. He has got, thereby, some ideas which differ materially from the conceptions generally held with regard to the geology of these republics. He regrets that he has paid only short visits to the republics — the longest having been an excursion to NW. Venezuela and Cucuta, where he travelled as a guest of the Caribbean Petroleum Company — but he is of the opinion that this lack of local experience needs not impede him to form and to publish personal conceptions on the geology of the area. There are especially three subjects which the author would like to treat: the age of the non-fossiliferous slates and sandstones and of the metamorphic schists, the areal distribution of the cretaceous, and the historic geology and the tectonics of the tertiary.

The metamorphics of Venezuelan Guyana, which so clearly belong to the Guyanan-Brazilian shield, are generally regarded as prepalaeozoic, although this age has not been proved with absolute certainty (45). In the following we shall not discuss them; we shall occupy ourselves exclusively with the metamorphics of the cordilleran region.

The slates and schists of the Cordilleras have been regarded by many authors as palaeozoic or pre-palaeozoic. W. BERGT (5, 6, 7) called the slates palaeozoic and the schists prepalaeozoic; W. SIEVERS (36, 37, 38) considers slates and schists both to be archaean; E. GROSSE (13), under the influence of G. STEINMANN, regards all the slates and schists of Antioquia as prepalaeozoic and thinks that their age increases with increasing metamorphism; E. HUBACH (17) and E. SCHEIBE (30) consider the Quetame-schists to the East of Bogotá as praecambrian and H. GERTH has, in the most general sense, pronounced that slates and schists in Northern South America are prepalaeozoic. He writes:

„Die Hauptmasse der fossilfreien regionalmetamorphen Gesteine, die durch die ganzen Kordilleren ...... immer wieder an die Oberfläche kommen, halte ich für algonkisch. Wiederholt sehen wir ...... paläozoische oder mesozoische Schichten mit den metamorphen Gesteinen verfaltet, ohne dass die erstere ihre Fossilführung eingebüsst haben ...... Wohl werden auch die jüngeren Sedimente bei intensiver Faltung dynamometamorph verändert, aber dann nehmen sie doch nicht den Habitus der hier behandelten regionalmetamorphen Gesteine an" (11, p. 80—91).

Two years later, GERTH is somewhat more prudent in his opinion, at least with regard to the schists of the Caribbean Coast Range, where LAMARE has found — amidst the schists — true radiolarites (24). GERTH writes:

„Wenn diese (the radiolarites) auch nicht gerade als absolut beweisend für ein paläozoisches Alter der betreffenden Gesteine angesehen werden können, möchten wir uns nicht mehr so entschieden gegen das paläozoische Alter gewisser metamorpher Gesteine in der nördlichen Kette des karibischen Gebirges aussprechen, wie wir es früher getan haben" (12, p. 346).

Whereas the authors hitherto mentioned apparently try to give the rocks in question a very high age, there are others who have regarded them as palaeozoic or, partly, even as mesozoic. EUGSTER (10) calls the metamorphics of the Cordillera de Bogotá palaeozoic; R. SCHEIBE (32) equally considers some schists of the Cordillera Central of Colombia as palaeozoic; DE CIZANCOURT and SCHNEEGANS (9) do the same with respect to the Venezuelan metamorphics. REINHARD (28), who is strongly convinced of the postalgonkian age of the schists of the Coast-Range in NE. Venezuela, discusses the possibility of their mesozoic age.

Various authors, who described slates and schists from Colombia and Venezuela, have not given an opinion about their age. Among these we may mention AUBERT DE LA RÜE (3), BENDRAT (4), HETTNER–LINCK (16), RUTTEN (29) and STUTZER (40).

In general, the hitherto mentioned authors did not give thorough arguments for their opinions; we may say that their opinion has been based chiefly on a certain "feeling". Two arguments, however, have been put forward from time to time. Firstly that there are known from Colombia and Venezuela non-metamorphic, fossiliferous, carboniferous, devonian, and, according to some geologists, non-metamorphic ordovician deposits, and that therefore all the metamorphics must be predevonian resp. pre-ordovician. Secondly that the lack of fossils in the slates and schists under discussion proves their great age. It is desirable to argue that these two "proofs" have only very small value. As to the first argument we may cite that, on the island of Aruba (44) there have been found hornblendeschists of cretaceous age which are strongly metamorphic and which, according to their metamorphism would f.i. be regarded by GROSSE (13) as to be of archaean age. Moreover, there occur in many countries at the side of non-metamorphic, older rocks younger ones which are more or less metamorphic. As to the second argument it is sufficient to indicate that f.i. the Cambrian of the Ardennes and the Silurian of Brabant are slightly metamorphic and that they are so poorly fossiliferous, that, if they occurred in a country so slightly investigated as Colombia or Venezuela, they certainly would not have yielded fossils until now.

We shall now pass to the discussion of the different regions, for which there have been given somewhat better arguments for the age of their metamorphics.

1. W. KEHRER (22) found in the southernmost Colombia two metamorphic series: "Glanzschiefer", phyllitic slates and quartzitic sandstones are resting with an angular unconformity on sericite-chlorite-schists, phyllites and quartzitic conglomerates. He proves with these interesting observations that there exist certainly two independent formations of schists in this part of the country, of which the older one presents the strongest metamorphism. He regards the older formation as Algonkian and the younger one as possibly Cretaceous, but for this interpretation he does not give convincing arguments. As a matter of fact the algonkian age remains absolutely unproved, as palaeozoic rocks in this part of the country have not been found until now.

2. R. SCHEIBE (31) has found in Antioquia a formation of porphyritic lava's and tuffs, the cretaceous age of which has been proved by some intercalated fossiliferous layers. In sections, where this porphyritical formation outcrops, he finds also a series of green schists, and he feels sure that these schists are of the same age. E. GROSSE (13) regarded some years later, on the base of their metamorphism, these schists as of middle-algonkian age. I should say that SCHEIBE's opinion is by far more convincing, but that he has n o t p r o v e d the cretaceous age of his green schists. As he has not shown — by careful sampling and by microscopical study of the samples — that there exists a gradual transition between the members of the porphyritic formation and the green schists, it remains possible that there are two formations of quite different age, which, in consequence of intense folding or thrusting, have been brought into seemingly normal contact.

3. HETTNER (15) described from the area E. of Bogotá a series of blue clayshales, quartzites and quartzitic conglomerates which he called "Quetame-series" and which he regarded as probably pre-cretaceous: they occur below fossiliferous, cretaceous rocks. STUTZER found to the W. of this zone of ancient rocks (40) fossiliferous, non-metamorphic carboniferous strata, which contain, however, according to W. KEHRER (21) true quartzites. In the neighbourhood have been found also non-metamorphic, devonian rocks (33). The Quetame schists are, therefore, very probably pre-devonian. W. KEHRER (22) calls them even pre-palaeozoic, and, although this is quite possible, it is far from being proved. We have f.i. to consider that there have been found in Colombia ordovician fossils in a series of rocks, which, according to SCHUCHERT (33), are "strongly folded quartzites, slates, marbles and black limestone", whilst, in Venezuela, ordovician fossils have been found in "a partially metamorphic shale and sandstone series, in some places approaching a micaceous schist" (TERRY, teste SCHUCHERT, 33, p. 693). L. KEHRER, equally, calls the rocks of the Venezuelan Ordovician "semimetamorphic" (20). It is, therefore, very well possible that the semimetamorphic Quetame-schists are silurian.

4. The strongly metamorphic rocks of the Eastern Cordillera N. of Bogotá may very well be pre-palaeozoic. It is, however, necessary to

consider that there have been found in the Cordilleras during the last twenty years more and more outcrops of "hercynian" rocks, and that it is not impossible that the strongly granitized rocks N. of Bogotá are palaeozoic, and metamorphosed by hercynian granites. LLERAS CODAZZI (26) even regards the granites of this region — without proving his opinion — as "Andengesteine", i.e. as old-tertiary or as youngest-cretaceous.

5. The silver-white phyllites which have been found as float in rivers on the eastern side of the Sierra de Perijá (25) are very probably pre-devonian, because in these same rivers the devonian is fossiliferous and non-metamorphic. It can, however, not been proved for the present that these phyllites are prepalaeozoic.

6. The gneisses of the Sierra Nevada de Santa Marta may very well be pre-palaeozoic, but they might equally well be palaeozoic, their metamorphism being due to hercynic granitization or to hercynic dynamometamorphism (5, 36).

7. The schists of Goajira peninsula (40) are certainly pre-cretaceous, their pebbles having been found in cretaceous conglomerates; it is, however, impossible to say, even approximately, how much older than the Cretaceous they are.

8. The area in Venezuela where the metamorphics have been best studied, is the Cordillera de Merida. In the past thirteen years there have been published four stratigraphical surveys, which are reproduced here in a diagram. This diagram shows, that great differences in interpretation are

Pre-mesozoic Stratigraphy of the Cordillera de Merida.

Christ 1927.	Oppenheim 1937.	L.Kehrer 1937 and 1938.	Kündig 1938.
	(Mucupati,post-palaeoz.)	(Mucupati,lower cretac.)	
Palmarito series, carboniferous.	Palmarito.	Palmarito.	Palmarito.
Mucupati series, (?Devonian limest. ,sandst., marl a. quartzite).		Mucuchachi series, (?Devonian: quartz-slates to phyllite etc. Rare traces of fossils)	Mucupati ? (Devonian?)
Caparro-Bellavista-series (?lower palaeoz.limest., shales,sandst., micaschists).	Caparro-Bellavista ser. (Middle ordovician schists and semimetamorphic quartzites).	Caparro-Bellavistá- (Miudle Ordovician ; metamorphic sandstones and shales).	Caparro series (Ordovician).
	Mucuchíes series (green -greyish and black schists and slates : Lower Ordovician).		
	Mucuchachi series (cambrian).		Ëo-palaeozoic granites in sills and apophyses.
	Micaschists and Estanque gneiss (Algonkian). Santo Domingo gneiss, quartzites,phyllites, old granites (Algonkian).		Eo-palaeozoic metamorphics (Mucuchachi ,Merida, Bellavista, Micaschists, Contact rocks.
Mucuchachi series (Archaean black schists)			Iglesias series. (Archaean gneisses, migmatites,arterites etc.

still in existence. There are only two pre-mesozoic formations, the Palmarito series and the Caparro series, as to which no disaccord exists among the four authors, their age having been proved by carboniferous and ordovician fossils. The Bellavista series, united by three authors with the Caparro, is separated from it by KÜNDIG (23) and regarded as older. The Mucuchachi phyllites are taken by CHRIST (8) for Archaean, by OPPENHEIM (27) for Cambrian, by KÜNDIG (23) for eo-palaeozoic. The fact that L. KEHRER (20) has found traces of fossils in this series — he mentions even the occurrence of ammonites — renders a pre-cambrian age highly improbable. KÜNDIG states that the rocks of the Mucuchachi series pass gradually into those of the Bellavista series, into micaschists and into true contactrocks on approaching a contact with post-Mucuchachi granites. It is clear that, if this is true, it becomes improbable that the Bellavista schists and the micaschists are pre-palaeozoic. As possibly pre-palaeozoic rocks there remain then only the rocks of the Iglesias series (KÜNDIG): gneisses, migmatites and arterites. KÜNDIG regards them as archaean "Unterbau-Gesteine". He does not prove, however, that the Mucuchachi-Bellavista-Micaschists-series and the Iglesias-series really belong to different geologic cycles. His sections nowhere show a normal contact between the rocks of the first and second group, nor does he mention the existence of conglomerates in the Mucuchachi, containing pebbles of the Iglesias series. As long as no unconformity has been found between the Iglesias and the Mucuchachi series nor pebbles of Iglesias rocks in Mucuchachi conglomerates, we must take into consideration the possibility that both series have the same (?palaeozoic) age, the rocks of the first series having been metamorphosed more intensely by stronger granitization and migmatization in deeper zones of the palaeozoic geosyncline.

9. LIDDLE has tried to prove that a great part of the schists of the Coast Range of Venezuela is of silurian age (25). His arguments, however, are by no means convincing, as he departs from the wrong supposition that limestones of the Coast Range have yielded silurian fossils (29).

10. LIDDLE has been the first to publish data on the occurrence of metamorphic, cretaceous rocks in Northern Venezuela (25) between the meridian of Barquisimeto and the eastern part of the Coast Range. He states that in this area cretaceous sandstones, shales and limestones have been converted into quartzites, schists and marbles. His contention that the metamorphosis is due to the influence of an intrusion of augite-porphyrite (25, p. 68—69) is certainly wrong. SCHÜRMANN (34) equally states that in the neighbourhood of Barquisimeto even the eocene rocks show signs of beginning metamorphosis, and that rocks of cretaceous age have been changed into silvery-white micaschists. L. KEHRER (19) says that the cretaceous limestones in NE and E. Lara have been changed into marbles, and the shales, marls and shales with organic substance into sericite-schists and graphite-schists. The statements of these three independent and

competent geologists are so positive that it seems impossible to avoid the conclusion that in this part of Venezuela there occur on a large scale metamorphics of cretaceous age. This conclusion, however, is so important that we can not feel satisfied by the general statements, published until now. Only a map, on which the transition from normal to metamorphic sediments in the direction of the strike of one or more structures is indicated, accompanied by a detailed description and by microphotographs of samples from crucial localities might satisfy completely our curiosity. It may be added that the author is not the only one who is longing for more evidence in this important matter. GERTH (11) has cited with some suspicion LIDDLE's opinion about the cretaceous metamorphics, and, in the second volume of his "Geologie Südamerikas" (12, p. 347), when SCHÜRMANN's publication had already appeared, he still regards the schists of the Serranía del Interiór as parts of the basal complex of Venezuela.

I may add, that, during an excursion in 1930, I collected myself some cretaceous rocks in Venezuela. The description of three samples follows here:

Lower Cretaceous Tomon Sandstone from Agua Caliente near San Antonio, SW. Venezuela. The rock is a white, somewhat porous, quartzitic sandstone. The slide presents in some places a very course, quartzitic structure. In other parts of the slide the rounded, original quartzgrains are still visible, each quartzgrain having, however, been enlarged by apposition, so that the general structure has become equally quartzitic. There are also places, where the pores between the original, rounded quartz grains have been filled with quartz which has an orientation of its own.

Lower Cretaceous Tomon Sandstone from La Cuchilla between Carora and Valera. A grey, quartzitic rock. The slide shows quartzitic structure throughout. In some places, however, the original "sand-grains" are still recognizable. Between the quartzes there are to be found many small flakes of sericite, sometimes passing into semi-coarse muscovite; moreover there occur some idiomorphic crystals of tourmaline. The rock is a sericite-quartzite with tourmaline.

Lower Cretaceous Tomon Sandstone from La Cuchilla between Carora and Valera. A red-yellow mottled, feebly schistous rock. The slide shows an irregular fabric of quartz grains and sericite flakes; the latter sometimes passing into muscovite. Small grains of limonite are responsible for the red colour. The rock is a slaty quartz-sericite schist.

From the foregoing description it is to be seen that the Cretaceous presents traces of metamorphism, even far to the W. of Barquisimeto. The described samples may be reckoned to belong to the uppermost part of the epizone. These facts seem to support the views of LIDDLE, SCHÜRMANN and L. KEHRER.

11. If I rightly understand the data of the geologic literature, the rocks, treated-with sub 10 all belong to the Serranía del Interior. SCHÜRMANN (35) as well as L. KEHRER (19) are of the opinion, that there occur equally cretaceous metamorphics in the northernmost range of the Caribbean mountains, and, according to LIDDLE (25, p. 75), there are to be found on the island of Margarita at the side of "silurian" schists also cretaceous metamorphics. As LIDDLE himself states that it is hardly possible to

differentiate these two complexes of schists, we may neglect for the moment his contentions.

The arguments of SCHÜRMANN and L. KEHRER are the following. The radiolarite described by LAMARE (24) from the northern part of the Coast Range might very well be a cretaceous rock, the Cretaceous of Venezuela abounding in many localities with radiolarites; the marbles, so common in the northern parts of the Caribbean Mountains might be metamorphosed cretaceous limestones; the limestone of Valencia — from which LIDDLE has erroneously mentioned silurian fossils — might be a cretaceous limestone. As to the last argument the authors might find support in a very old publication of KARSTEN (18), who, in a diagrammatic section, indicates the Valencia-limestone as "Exogyrenkalk" and who, in his text, points to the similarity of this limestone with that of the Morros de San Juan — on the southern side of the Serranía del Interiór — in which, according to LIDDLE, Exogyras have been found. TRECHMANN's discovery (42, 43) of rocks with cretaceous fossils in the Northern Range of the island of Trinidad, and SPATH's description of tithonic ammonites from the same area (39), which seem to indicate that the metamorphic rocks of this range are cretaceous, might equally support the views of SCHÜRMANN and KEHRER. HEDBERG (14), mentioning TRECHMANN's discovery, has indicated, on the other hand, that it is difficult to understand that in the easternmost part of the Coast-Range of Venezuela there should occur at the side of strongly metamorphic, cretaceous rocks other rocks of the same age which are absolutely non-metamorphic. Dr. TRECHMANN was so kind as to provide me with some samples from the neighbourhood of his fossil-locality in NE. Trinidad. Their description follows here.

Conglomerate, W. of Jetty, Toco, NE. Trinidad. It consists of pebbles of more than 1 cm with a sandy cement. The elements of the conglomerate are chiefly quartz and quartzite. Some small pebbles of a limonitic rock which contains quartz-splinters and muscovite-flakes. The cement contains quartz, muscovite and some calcite.

Dark, calcareous shale, E. of Jetty, Toco, NE. Trinidad. Very dark rock with an unmistakeable echinid spine. The slide shows numerous organic remains (echinid fragments, calcified sponge spiculae etc.) in a brown, dense, isotropic "groundmass". There is no trace of metamorphism.

Dark, calcareous shale, W. of Jetty, Toco, NE. Trinidad. The slide shows remains of organisms in a dark "groundmass" with quartz splinters and mica flakes. There is no trace of metamorphism.

Conglomeratic sandstone, S. of Toco, NE. Trinidad. The slide reveals the presence of the following elements: 1. very much quartz, partly cataclastic, 2. finegrained quartzite to quartzschist, sometimes with muscovite, 3. grains of perthite, 4. grains of orthoclase, 4. coarse quartzite. The cementing material consists of much quartz and sericite. The rock contains metamorphic material and has itself not suffered from metamorphosis.

Black limestone E. of Jetty, Toco, NE. Trinidad. There is — in the slide — not a trace of metamorphism in this very finegrained limestone; it contains siliceous sponge spiculae, and grains and some crystals of pyrite.

Conglomeratic, calcareous sandstone, 4 miles S. of Jetty, Toco, NE. Trinidad. The elements of the conglomerate are: 1. microcline, 2. abundant quartz, partly cataclastic, 4. grains of an intergrowth of quartz and plagioklase, 4. quartzite with some muscovite, 5. very dusty plagioclase.

Apparently there occur in this part of Trinidad non-metamorphic limestones and grits, which contain pebbles of metamorphic rocks. It has been known for a long time, that there occur also in the Northern Range of Trinidad true metamorphics. It is, therefore, not impossible that, in N. Trinidad, we have to do with a folded basement with in-folded jurassic and cretaceous rocks.

AGUERREVERE and ZULOAGA (1, 2) have, in two very recent publications, described the Serranía del Interiór and the Coast Range s. str. They distinguish to the S. of Carácas two different groups, both metamorphic: the Carácas group and the Villa de Cura group. They are of the opinion that there occur many cretaceous rocks in the Serranía del Interiór. Their conclusions are, however, somewhat contradictory. In their first publication they say that the Carácas group is cretaceous, in the second that the Carácas group is cretaceous or older, the Villa de Cura group cretaceous or younger. With regard to the rocks of the Coast Range s. str. they say at some place that they might equally be cretaceous, at other places, that they are possibly archaean. It seems better, for the moment, to leave out of this discussion the work of AGUERREVERE and ZULOAGA.

Whilst many observations seem to indicate that probably a part of the metamorphics of the "Serranía del Interiór" is cretaceous and that possibly some of the metamorphics of the Coast Range s. str. may be cretaceous, there is one publication which indicates that there occur probably also much older elements in the Coast Range. IJZERMAN (45) has studied a series of granitodioritic rocks of the Venezuelan cordilleras and comes to the conclusion that:

"the granitodiorites of the Andean folding area reveal less constant features than those of the old core of South America ...... On the one hand they are characterized by greater variabilities, on the other hand many types cannot be differentiated from those belonging to the old core ...... It is a fact of more significance that the typical orthite and the primary epidote we met with in many countries of the old core, seem to be absent in the Andean folding area" (45, p. 492—493).

Now, LAMARE (24, p. 142) has described from the Silla de Carácas in the Coast Range s. str. a granite, which, according to his description can hardly be distinguished from the granites of the Old Core; it contains large microclines, primary epidote and orthite. This discovery indicates that there may exist in the Coast Range, at the side of possibly much younger rocks, very old, precambrian elements.

The facts, discussed in the foregoing, allow for the following conclusions.

1. The age of by far the most non-fossiliferous slates and schists in the Columbian-Venezuelan Cordilleras is unknown; it is not permitted to

speak — as has been done very frequently — of an extensive precambrian basement.

2. There is only one case, where the precambrian age of rocks is rather probable: the granite of the Silla de Carácas.

3. Many schists are certainly precretaceous (Goajira, Sierra de Perijá, Sierra Nevada de Santa Marta, Cordillera de Merida, Cordillera Oriental) and probably predevonian (Sierra de Perijá, Cordillera de Merida, Cordillera Oriental); it is quite possible that many of these schists are precambrian.

4. The fact that W. KEHRER (22) has found in S. Colombia two different formations of schists, separated by an unconformity, indicates the possibility that there exist in both republics complexes of metamorphics of very different age.

5. It is possible that the green schists of Antioquia are cretaceous.

6. It is very probable that in some parts of the Caribbean Mountains (s.l.) there exist metamorphics of cretaceous age. Every generalization, which would lead to the conception, that all the metamorphics of this area are to be regarded as Cretaceous, is to regard with the greatest suspicion. The granite of the Silla de Carácas, the data of HEDBERG for the NE. part of Venezuela and the data procured by the Northern Range of Trinidad may induce us to the greatest prudency.

I cannot deny that the harvest of this critical study on the metamorphics of Colombia and Venezuela is somewhat meagre. We have seen that there exists the greatest uncertainty about the age of these rocks. It is, however, better to know this uncertainty than to feel satisfied with a not-founded knowledge. Happily, the remedy for gaining better knowledge can be indicated. Careful fieldwork, combined with intensive sampling and laboratory-study of the samples will with great probability lead to more definite knowledge of the age of the rocks under discussion. It is chiefly the green schists of Antioquia, the metamorphics of the Serranía del Interiór, parts of the Coast Range and the Northern Range of Trinidad, which promise good results.

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Physiology. — *Muscle sounds of a single twitch.* By G. VAN RIJNBERK and H. D. BOUMAN.

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Introduction.

HERROUN and YEO have described for the first time that a single twitch produces a muscle sound (1885). They stimulated the muscles of the upper arm with single shocs and listened with a stethoscope.

With stimuli strong enough to give a clearly visible twitch, a low sound was heart comparable to the first heart sound.

BERNSTEIN in 1890 found (also in listening experiments) that a single shoc given to the gastrocnemic of the rabit produces a muscle sound. It was of an explosive caracter comparable to *p* or *t*, or the sound of lighting a gas lamp. The frequency could not well be determined because of the extreme shortness of the sound.

I. *Experiments on dogs muscles.*

Our experiments were performed on dogs. Motor nerves were stimulated with single condenser discharges. The sound has been recorded by the same technique that has been used in our experiments on voluntary contractions in man. Dog and experimenter are enclosed in the sound proof room of the laboratory, the amplifier and recording string oscillograph are outside this room.

We have used the gastrocnemic and some segments of the rectus abdominis muscle. In all experiments a muscle sound has been found. Experiments have been performed with the nerve intact and with the nerve cut. The results in both cases shows no differences. In some experiments the muscle (gastrocnemic) was left completely in situ, in other experiments the tendon has been cut and the muscle freed from its surroundings and placed on cotton wool. (The blood supply was not damaged by the operation.) The reason for this was the following. In some experiments on muscles in situ we were struck by the extreme regularity of the records, a regularity which was completely unknown in our previous records of voluntary contractions in man. The records had the caracter of being due to the tapping of some structure which can easily be set in vibration. The records in fact is those of an aftervibration. The question arose whether this vibrating structure is the muscle itself or the underlying bone.

The answer to this question is obvious from two records reproduced here. One is from a muscle in situ while the other is from a muscle isolated as much as possible from the bone but with the blood supply undamaged. The muscle was placed on cotton wool.

The difference between the curves is clear. The vibrational caracter has entirely disappeared by the isolation of the muscle. Therefore these experiments are the first records of the true muscle sound of a twitch, HERROUN an YEO's and BERNSTEIN's results have been due to the under-

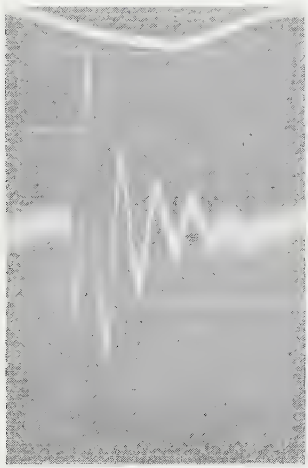

Fig. 1. Rec. E 20.
Muscle sound of a twitch
Muscle in situ.

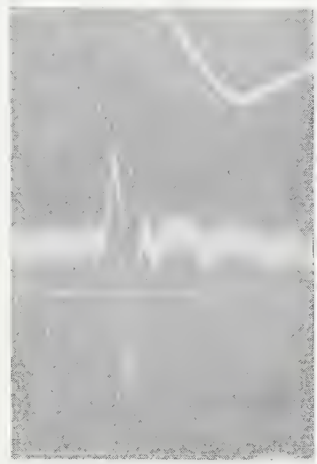

Fig. 2. Rec. E 82.
Muscle sound of a twitch, muscle isolated with undamaged bloodsupply and placed on cottonwool.

lying bones. A sound as recorded in fig. 1 can doubtlessly give an aural impression as described by BERNSTEIN. The sound recorded in fig. 2 would sound entirely different, if at all audible.

In all experiments the main feature of the muscle sound is a single diphasic effect: the $\alpha\beta$ effect. In many curves this is followed by some

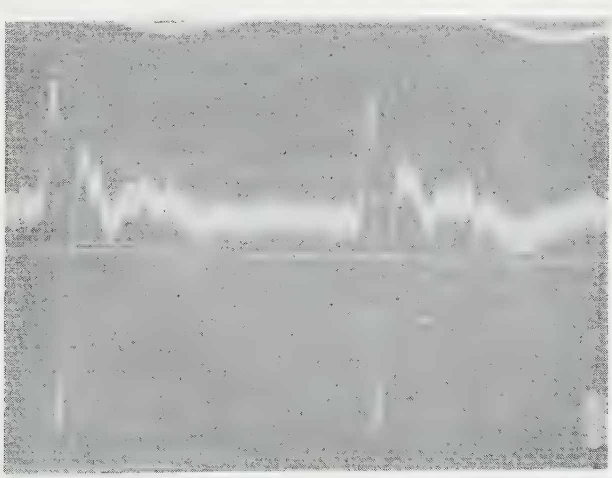

Fig. 3. Rec. 84.
The $\gamma\delta$ complex of the muscle sound of a single twitch of an isolated gastrocnemic.

Fig. 4. Rec. 48. Freq. 4 per sec.

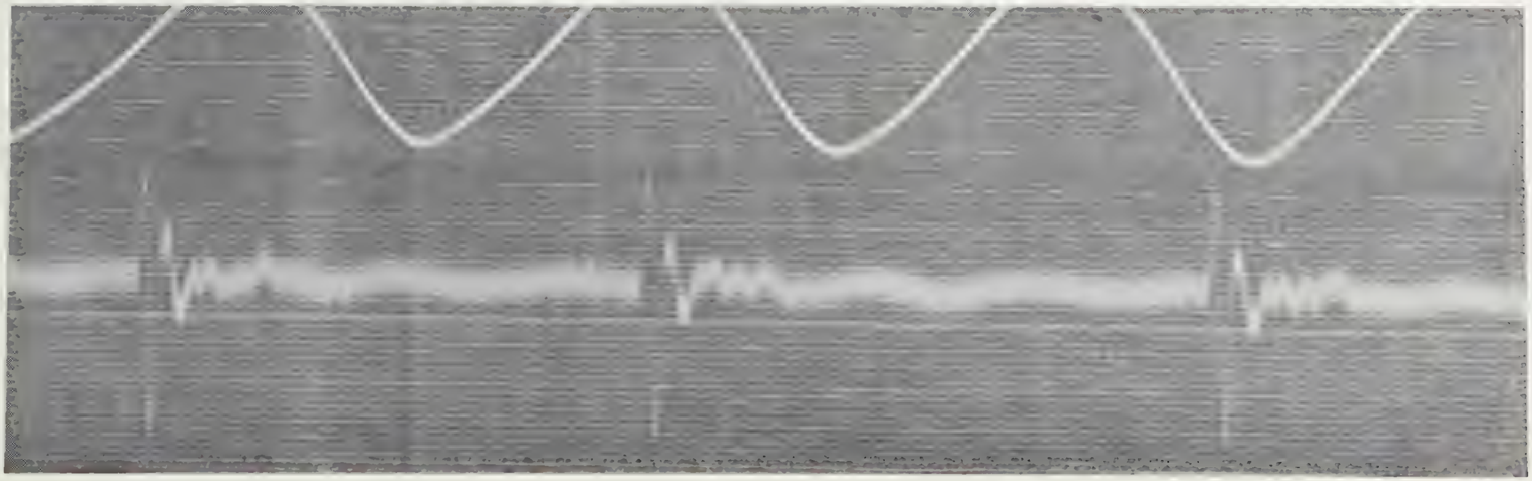

Fig. 5. Rec. 51. Freq. 8 per sec. Fig. 1—5 gastrocnemius. Time $^{1}/_{10}$ sec.

irregular vibrations of low amplitude, which are not comparable to after vibrations. In some records where they are exceptionally big they have been labelled $\gamma\,\delta$ complex (see fig. 3).

The curve does not change on cutting the motor nerve, the complex therefore cannot be described as a reflectory after discharge following the main twitch. As the nerve has always been stimulated close to the muscle, differences in conduction time of different nerve fibres cannot account for the after effect.

An increase in frequency of the stimuli gives little or no change in the recorded sounds. (See figs. 4 and 5.)

The phenomena recorded at higher frequencies when tetanic contraction sets in will be the subject of a seperate paper.

The experiments show therefore that a single twitch does produce a muscle sound. If the muscle is in situ above the underlying bone the caracter of the sound is determined mainly by the bone. It is comparable to after vibrations.

In an isolated muscle however the sound is much more simple. The main feature of the sound is a single diphasic vibration ($\alpha\,\beta$ complex) followed in many experiments by a second diphasic vibration of much smaller amplitude ($\gamma\,\delta$ complex). It is impossible to discuss the pitch of such a phenomenon.

We thought of finding a good muscle for these experiments in the rectus abdominis muscle of the dog. We thought it reasonable to suggest that if a middle segment of this muscle contracts, the surrounding segments will isolate it from its bony insertions (sternum, pelvis). The curve reproduced in fig. 6 shows that this is indeed possible.

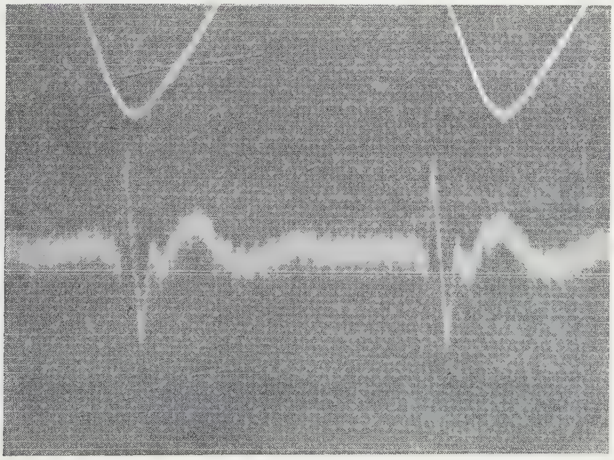

Fig. 6. Rec. 116.
Twitch of M3 of the rectus abdominis.

The necessity for great care however becomes obvious from a comparison

of this record with the record reproduced in fig. 7. The contracting segment was situated one segment more caudad, and therefore closer to the pelvis.

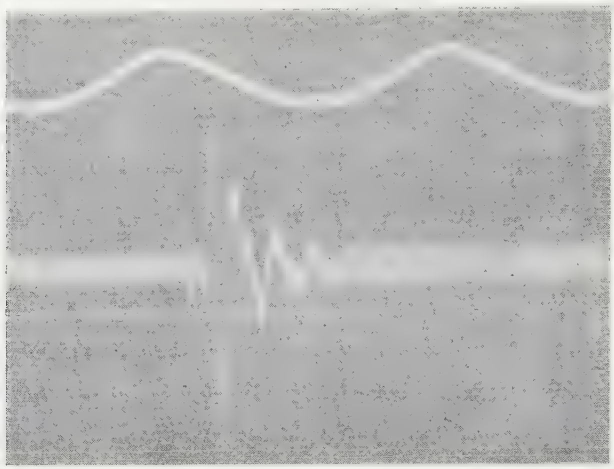

Fig. 7. Rec. 189.
Twitch of M4 rectus abdominis.

In this curve the regular vibrations of an after vibration can already be recognised.

The sharp excursion in this experiment preceding the complex is a signal which recorded the moment of the stimulus.

II. *Experiments on frogs.*

In order to verify the results obtained with dogs muscles, some experiments were performed on the more easily available gastrocnemic of the frog. Excised nerve muscle preparations have been used throughout.

A simple apparatus was constructed which enabled us to keep the muscle under any tension necessary and at the same time eliminated vibrations not due to the muscle itself. It consists of a heavy lead bloc. On one end a support was attached (also of lead) in which the femur could be clamped. A small lead carriage moving on ball bearings and provided with another clamp served to secure the tendon. For isotonic contractions a cord attached to the carriage could be loaded with different weights. For isometric contractions the carriage could be fastened to the supporting lead bloc.

Records of the acoustical phenomena of a single isometric twitch showed a constant picture, consisting in one single diphasic vibration. There is no after effect ($\gamma\ \delta$ complex). The results therefore appear to be in close agreement with those obtained on dog muscles.

Yet we do not expect to continue our experiments with frogs as the much stronger and bigger dog's muscles give much larger excursions in our records.

Conclusion.

The acoustic phenomenon of the muscle twitch suffers from serious interference from sounds produced by the bones if the muscle is stimulated in situ. The interference appears in the form of a common type of damped vibration. If the muscle is isolated from its bony insertions and placed on cotton wool (blood supply undamaged), records of a much simpler type are obtained.

The acoustical effect then consists of a single diphasic excursion (called $\alpha\beta$ complex) followed by another diphasic excursion of much smaller amplitude (called $\gamma\delta$ complex). Records obtained from frog's muscles are not essentially different from those obtained from dog's muscles.

Acoustics.— *The residue, a new component in subjective sound analysis.* By J. F. SCHOUTEN. (Natuurkundig Laboratorium der N.V. Philips' Gloeilampenfabrieken, Eindhoven, Holland.) (Communicated by Prof. G. HOLST.)

(Communicated at the meeting of February 24, 1940.)

§ 1. *Introduction.*

In a former paper [1]) we described some experiments performed with an optic siren which permits to realize periodic sounds of any prescribed wave form. It was found that the non-linear distortion in the human ear is by far less pronounced than is stated by some authors and in particular that if a complex sound of moderate loudness objectively misses the fundamental tone, this tone is *not* generated in any appreciable amount in the ear. It is a fact that a pitch equal to that of the fundamental tone is ascribed even to those sounds in which the fundamental tone is not present. The almost generally accepted hypothesis to account for that behaviour consists in the assumption that this fundamental tone is generated within the ear by means of non-linear distortion. Our experiments thus proved this hypothesis to be invalid.

We were finally led to speculations as to how the ear might ascribe a pitch to a complex sound and suggested that the periodicity of the wave form rather than the distance of the harmonics in the Fourier spectrum might be the physical property determining this pitch. No mechanism presenting itself to enable the ear to perceive that periodicity, the question was left there unsolved.

Further investigations and in particular the studying and repeating of experiments almost a century old [2]) led us to the conclusion that the fundamental problem underlying these and similar paradoxical phenomena is not a question of perception of pitch, but rather a question of subjective sound analysis.

Once a radical change is made in OHM's seemingly trivial acoustical law of sound analysis [3]) the explanation of the "case of the missing fundamental" [4]) and similar problems follows quite naturally.

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1) J. F. SCHOUTEN, The perception of subjective tones. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **41**, 1086 (1938).

2) A. SEEBECK, Beobachtungen über einige Bedingungen der Entstehung von Töne. Pogg. Ann. **53**, 417 (1841). A. SEEBECK, Ueber die Sirene. Pogg. Ann. **60**, 451 (1843).

3) G. S. OHM, Ueber die Definition des Tones, usw. Pogg. Ann. **59**, 513 (1843). G. S. OHM, Noch ein Paar Worte über die Definition des Tones. Pogg. Ann. **62**, 1 (1844).

4) S. S. STEVENS and H. DAVIS, Hearing, New York 1938, p. 99.

This extension of OHM's law involves an important consequence both as regards our conceptions of subjective sound analysis, as well as regards those of the mechanism of sound perception.

We shall, therefore, although the essential clues for solving the problem can be found on page 1092 of our former paper, reintroduce the matter from the very beginning.

§ 2. *Subjective analysis of a periodic impulse.*

Periodic sounds containing a great number of higher harmonics present themselves to the untrained ear as one sound of a certain sharp tone quality with a pitch equal to that of the fundamental tone. Since HELMHOLTZ' careful investigations [5]) we know that the suitably trained ear is able to perceive the lowest dozen of harmonics with ease separately in the sound. This confirms OHM's acoustical law which states that sinusoidal vibrations only are perceived as a pure tone, that a complex sound is analysed by the ear into its different sinusoidal components and that these components will be perceived as pure tones having a pitch determined by the respective frequencies.

As a first experiment we listen to a periodic impulse of width $\frac{2\pi}{20}$ [6]) which contains the lowest score of harmonics in slowly decreasing amplitude (Fig. 1*a*). The fundamental frequency used was 200 cycles/sec. Our conclusions to be made are naturally restricted in that respect.

In this periodic impulse a strong and sharp note of pitch 200 is imme-

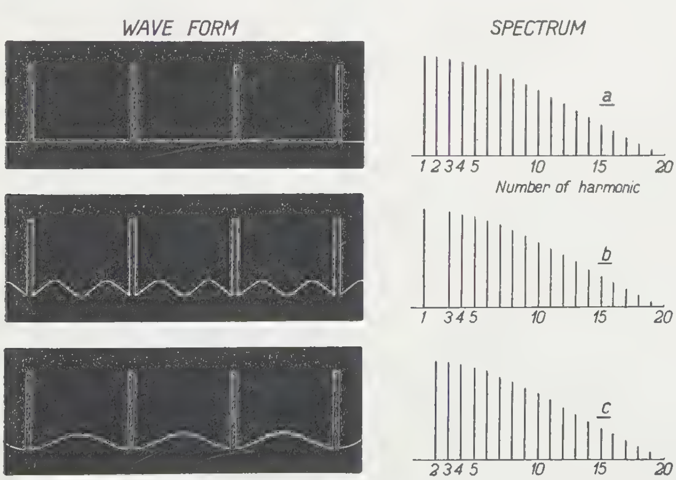

Fig. 1. *Elimination of components. a.* Periodic impulse of width $\frac{2\pi}{20}$. *b.* Periodic impulse without second harmonic. *c.* Periodic impulse without fundamental tone.

[5]) H. HELMHOLTZ, Die Lehre von den Tonempfindungen, 1862, Kapitel IV.

[6]) J. F. SCHOUTEN, l.c. p. 1090.

diately perceived, whereas some higher harmonics, although by far weaker than this gravest note, can be heard separately one after another if the attention is fixed upon them.

A very critical method of drawing the attention even of an untrained listener to a particular harmonic consists in adding first (by means of the second holder of the optic siren) that harmonic in the same amplitude but in opposite phase to the impulse. The harmonic, e.g. the second, (Fig. 1*b*) is thus cancelled from the sound. By then covering the second holder the harmonic is made to reappear and can be heard in often surprising clearness. The loudness of the harmonics decreases rapidly with increasing order. The twelfth harmonic is about the last one which can be heard separately, the higher harmonics are *not* separately perceptible.

So far the experiments, but for the great loudness of the gravest note (generally identified with the fundamental tone), present nothing essentially new. If, however, we now cancel the fundamental tone (Fig. 1*c*) we find that the sharp note of pitch 200 remains unchanged present in the perceived sound. Moreover, if thereupon the fundamental tone is again added to the sound, *it is heard separately as a pure tone of pitch 200* of low loudness comparable to that of the second and third harmonic. After some training this fundamental tone may even be heard without any help, although with more difficulty than the next harmonics.

The crucial point is thus that, as to subjective analysis, the sound contains *two* components of pitch 200, one of which, having a pure tone-quality is identical with the fundamental tone, whereas the other, having a sharp tone quality and great loudness, is of different origin.

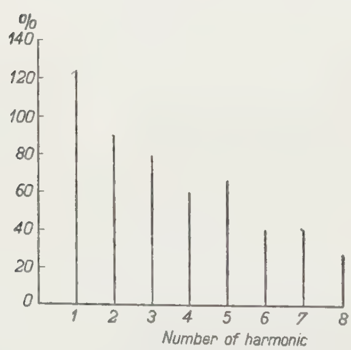

Fig. 2.
Loudness of subjective pure components (compare Fig. 1*a*).

Measurements of the loudness of the various harmonics were taken by comparing in successive contrast a particular harmonic in the periodic impulse and a pure tone of the same frequency. In Fig. 2 the relative amplitudes of the test frequencies, necessary for the match, are given. It will be seen that the fundamental tone in the periodic impulse is heard slightly too strong, the second harmonic in almost its true loudness and the higher harmonics gradually weaker and weaker.

If the harmonics are eliminated one by one, starting from the lowest, the sharp note does not, at first, materially change either in character or in loudness. If, however, the same is done starting from the highest the sharp note gradually loses in sharpness as well as in loudness. This behaviour suggests that the sharp note is connected with the presence of *high* harmonics in the complex sound.

If the intensity is raised, the pitch of the sharp note remains practically unchanged. The fundamental tone, however, exhibits the well-known fall up to about half a tone.

The totally different nature of the fundamental tone and the sharp note of pitch 200 is best brought out by adding the fundamental tone in increasing amplitude to the impulse. The increasing loudness of the fundamental tone is easily perceived, whereas the sharp note does not materially change in character and merely seems slightly to *decrease* in loudness.

Thus the fact that the lowest note in a complex sound is more easily perceptible than e.g. the second, third and fourth harmonic is not due to a particular enhancement of the loudness of the fundamental tone or to a general impairment of the harmonics in question, but to the presence of a hitherto unknown additional subjective component [7]) of almost the same pitch as the fundamental tone.

§ 3. *The hypothesis of the residue.*

Hitherto, as expressed in OHM's law, it was generally accepted that the ear analyses a complex sound into components of a pure tone-quality each of which corresponds with one frequency of the inner-ear sound field [8]). The difficulty remained that the highest harmonics, although not separately present in subjective analysis, add materially to the loudness as well as to the tone-quality of the complex sound.

We now find that, apart from those pure components, an additional component of sharp tone-quality may exist which cannot be correlated with any single frequency of the sound field.

We propose to call such an additional subjective component a "*residue*". It is very well possible that in a complex sound several residues are present.

OHM's law of subjective sound-analysis may now be extended as follows:

1. *The ear analyses a complex sound into a number of components each of which is separately perceptible.*

2. *A number of these components corresponds with the sinusoidal oscillations present in the inner-ear sound field. These components have a pure tone-quality.*

3. *Moreover, one or more components may be perceived which do not correspond with any individual sinusoidal oscillation, but which are a collective manifestation of some of those oscillations which are not or scarcely individually perceptible. These components (residues) have an impure, sharp tone-quality.*

§ 4. *The pitch of the residue.*

The loudness of the residue in a periodic impulse is greatly diminished

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[7]) The alternative hypothesis, consisting in the supposition that the observed sharp note is not a *component* of the sound, but the *total sound itself* leads to a great many difficulties and can, we believe, be discarded.

[8]) Thus including the components generated within the ear by non-linear distortion.

if the higher harmonics are cut off, e.g. by means of a low-pass filter. The residue is thus a collective manifestation of those higher harmonics.

Which physical property of these harmonics might determine the pitch of the residue? Two possibilities present themselves: either the distance between the harmonics, or the periodicity of the total wave form of the harmonics in question. In the case of the periodic impulse the two properties lead to the same pitch, the distance between the harmonics and the periodicity both being 200 cycles/sec.

Comparison of the two wave forms represented in Fig. 3 enables us to answer this question. The first wave form contains the even harmonics

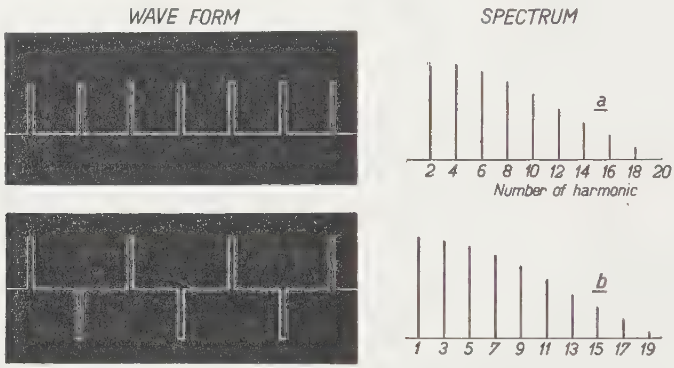

Fig. 3. *Two wave forms having the same distance between the harmonics but a different periodicity.*

only and is thus the octave of the periodic impulse used hitherto. The distance between the harmonics is 400 cycles/sec., the periodicity of the wave form, as well as of any group of adjoining harmonics is also 400 cycles/sec. The pitch of the residue is found to be 400. The second wave form contains the odd harmonics only. In this case the distance between the harmonics is again 400 cycles/sec., the periodicity, however, be it of the total wave form or of any particular group of adjoining harmonics is 200 cycles/sec. We found that a residue is present, although less pronounced than for the first wave form, and that the pitch of this residue is 200. No indication of any subjective component of pitch 400 was found.

The ear thus ascribes a pitch to a residue by virtue of the periodicity of the total wave form of the harmonics which are responsible for this residue.

Summarizing we are thus led to the following view upon subjective analysis of a periodic sound containing a great number of harmonics. The lower harmonics can be perceived individually and have almost the same pitch as when sounded separately. The higher harmonics, however, cannot be perceived separately but are perceived collectively as one component (the residue) with a pitch determined by the periodicity of the collective wave form, which is equal to that of the fundamental tone. We are thus

confronted with the surprising result that the harmonics *highest* in frequency, are perceived as a subjective component almost *lowest* in pitch. In a periodic impulse the residue must be the most prominent component due to its richness in harmonics and to the comparatively high sensitivity of the ear to these harmonics.

The pitch of a complex sound now follows quite naturally: *The pitch ascribed to a complex sound is the pitch of that component to which the attention, either by virtue of its loudness, or of its contrast with former sounds is strongest drawn.*

Therefore the pitch of a complex sound may be different depending upon the circumstances under which it is heard. An example of this behaviour was given on page 1092 of our former paper.

§ 5. SEEBECK's *experiments.*

It should be definitely stated that the views proposed here, although rather radical in comparison with the modern acoustical point of view, are no more than an extension of the work of the admirable and ununderstood SEEBECK. In 1841 SEEBECK (l.c.) described a great number of experiments performed with his acoustic siren. This lead OHM (l.c.) in 1843 to the formulation of his famous law. Far from being satisfied SEEBECK in the same year published a second paper in which he proved that OHM's law, although qualitatively accounting for the observed phenomena, could by far not describe these quantitatively. Therefore SEEBECK proposed an extension of OHM's law. The discussion is closed by OHM in 1844 with a second paper, which, though humorous, added nothing essential to clear the difficulties.

HELMHOLTZ (l.c.) too, while discussing the controversy between SEEBECK and OHM impresses the opinion upon the reader that SEEBECK, though otherwise a very keen observer, must have been mistaken here.

We shall describe three of SEEBECK's experiments. In the first experiment air was blown from a pipe against a revolving disk containing a number of concentrically arranged equidistant holes. SEEBECK thus produced a "periodic impulse". If a second pipe was placed against the same side of the disk at half-interval distance from the first the pitch was heard to jump an octave upwards (Figs. 5*a* and 5*b*). The number of impulses per second is thus doubled, the spectrum will contain the even harmonics in double amplitude and no odd harmonics. SEEBECK's objection to this formulation as a means of quantitatively describing the observed phenomena is the following: in the first sound the loudness of the gravest note is very great compared to that of the second and third harmonic. If the odd harmonics are compensated and the even ones doubled, the second harmonic (now the fundamental tone of the new sound) should be a little stronger than in the first sound but not so strikingly strong as is actually heard.

HELMHOLTZ suggests that SEEBECK, while not using the proper means

of drawing his attention to the second harmonic of the first sound, may have underrated its loudness. The experiment, however, is easily repeated with our optic siren and completely confirms SEEBECK's description. In view of the hypothesis of the residue the explanation is simple. The strong gravest note heard by SEEBECK is not the fundamental tone (which must have been of almost the same loudness as the second harmonic) but the residue. Doubling the frequency of the impulse not only doubles the amplitude of the second harmonic but *also* shifts the residue an octave upwards, thereby seemingly increasing the loudness of the second harmonic.

SEEBECK's second experiment is more immediately convincing. He compares three wave forms (Fig. 4). The first (Fig. 4*a*) is again obtained by one pipe. The second (Fig. 4*b*) by two pipes at a distance equal to the

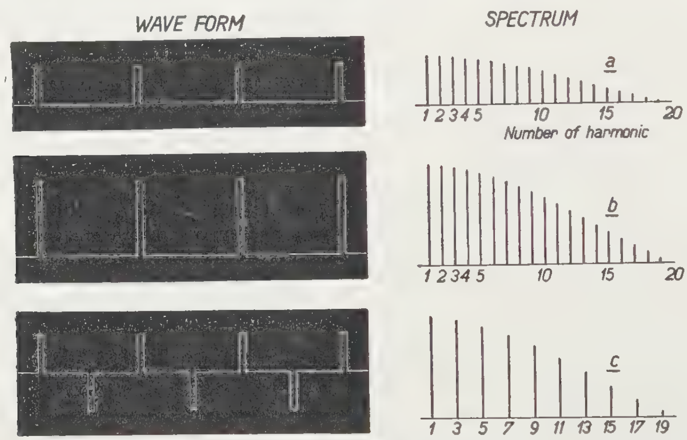

Fig. 4. SEEBECK's *experiment 2*. The amplitude of the fundamental tone in *b* and *c* is twice that in *a*. Yet as to subjective analysis the gravest note in *c* is barely stronger than in *a* and strikingly weaker than in *b*.

interval of the holes. This merely results in doubling the amplitude of the periodic impulse. The third wave form (Fig. 4*c*) is obtained by placing the second pipe against the opposite side of the disk at half-interval distance from the first pipe.

It is easily verified that the amplitude of the fundamental tone (which has the same pitch for the three wave forms) in the second and third wave form is twice that in the first one.

Contrary to OHM's expectation SEEBECK finds that in the third wave form the gravest note is barely stronger than in the first and strikingly weaker than in the second wave form.

Again the explanation is simple once it is realized that SEEBECK did not observe the fundamental tone, which must have been comparatively weak, but the residue. This residue must be much weaker in the third than in the second wave form since the former contains only half the number of frequencies.

Lastly we describe a third very elegant experiment. If, in the case of

two pipes placed against the same side at half-interval distance, this distance is slightly changed, the pitch of the sound immediately jumps an octave downwards. For demonstration of this effect SEEBECK constructed a disk containing four concentric rows of holes. In the first row the distance between the holes was 20°, in the second 10°, in the third alternately $9\frac{1}{2}$° and $10\frac{1}{2}$° and in the fourth 9° and 11°.

The second row, of course, gave a pitch an octave above that of the first row, in the third the octave was still the most prominent although the lower note was distinctly audible, in the fourth this lower note was more prominent than the octave. It is seen from Fig. 5 that even for the last wave form the fundamental tone is still very weak.

We repeated this experiment by putting one impulse in the first, another

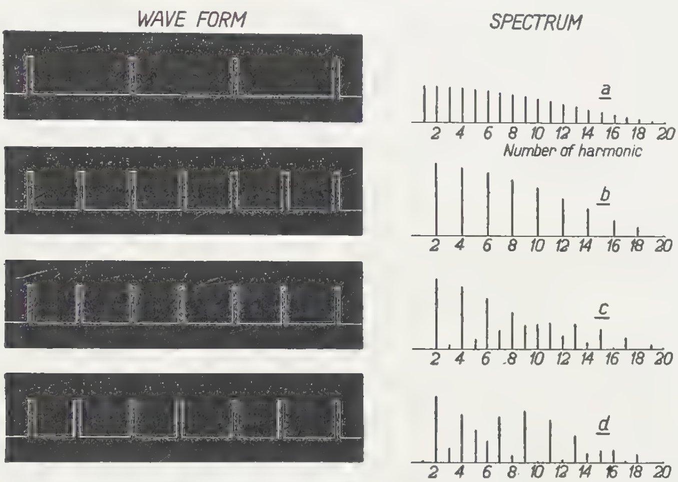

Fig. 5. SEEBECK's *experiment 3.* The pitch of wave form *b* (400) is an octave above that of wave form *a* (200). In wave form *c* a component of pitch 200 is distinctly audible. In wave form *d* it is even more prominent than the component of pitch 400. The fundamental tone, however, is very weak in wave form *c* as well as in wave form *d.*

in the second (turnable) holder of the optic siren. If the two impulses were exactly half a period apart the pitch was 400. A slight turning of the second holder, however, was sufficient to make the pitch drop to 200. The effect is very striking indeed. Using again the impulse of width $\frac{2\pi}{20}$ we observed the same behaviour as that described by SEEBECK.

In terms of the residue the explanation of this effect is that this slight change causes especially the higher odd harmonics to reappear which results in the change of pitch of the residue from 400 to 200. The fundamental tone itself does not play any part in the phenomenon and, in fact, can be heard very weakly after a slight change and more strongly after a great change in the distance between the two impulses, quite in accordance with its objective intensity.

To account for these phenomena, all of which show a discrepancy between the theoretical and experimental loudness of the gravest note in subjective analysis, SEEBECK suggested that *those higher harmonics, which cannot be perceived separately and which have a common period equal to that of the fundamental tone, in some way or other enhance the loudness of this fundamental tone. He even went so far as to suggest that this enhancement is not merely due to non-linear distortion in the ear* (l.c. 1843, p. 480).

The essential feature of our investigations consists in stating that it is not the fundamental tone itself which is so much stronger than should be expected, but that an additional subjective component of almost identical pitch and of often great loudness is present in the sound.

It is OHM's merit to have indicated the general principle underlying the phenomena of subjective tone analysis and to have formulated a law which up till now was considered to be so trivial that a renewed critical testing seemed superfluous.

We cannot but immensely admire SEEBECK to have realized, by means so simple as his acoustic siren the short-comings of OHM's law and even to have suggested an extension of this law which was scarcely considered seriously until now, almost a century later.

In a following paper we hope to investigate the theoretical consequences of the existence of the residue and to show that, as regards the physical part of the analysing mechanism in the inner-ear, the existence of this new component is by no means so improbable as one might expect beforehand.

§ 6. *Additional remarks.*

It is of interest to reconsider now some other older and newer experiments on subjective sound analysis.

KÖNIG [9]), experimenting with SAVART's siren, found that while holding a stiff wooden peg against the cogwheel (128 teeth, time of revolution 1 sec.) *a rattling sound as well as the tone c* (128 cycles/sec.) *are heard.* If however, the edge of a paper card is taken, the rattling sound is scarcely perceptible whereas the tone *c* can be heard very clearly.

Our interpretation of this behaviour is that the first sound contains a great many harmonics and thus brings about a residue (the rattling sound) whereas the second sound is comparatively pure.

A highly remarkable consideration can be found in STUMPF's work. STUMPF [10]) experimenting on synthetic vowels obtained by sounding together various harmonics of a given frequency finds that the pitch is equal to that of the fundamental tone even if that tone is not objectively present. He ascribes this throughout to a difference tone generated within

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[9]) R. KÖNIG, Ueber den Zusammenklang zweier Töne. Pogg. Ann. **157**, 226 (1876).
[10]) C. STUMPF, Die Sprachlaute. Berlin 1926, p. 185.

the ear by non-linear distortion. At one place, however, he wonders: "Yet I have doubted sometimes whether it (the difference tone) is, even subjectively, really present within the ear. One might imagine that in a sound consisting of the objective tones $c^2g^2c^3e^3$, *the pitch* c^1 *only* presents itself to the listener, without the fundamental tone c^1 entering subjectively into the sound".

This paradoxical formulation is very similar to our interpretation in our former paper, which we now, however, prefer to abandon in favour of the supposition that an alien component, the residue, is present in the sound. This component may, if loud enough, determine the pitch of the total sound.

The hypothesis of the residue may also be of importance in connection with the *strike note of bells*. We hope to deal more explicitly with that fascinating problem later. It may suffice now to draw attention to its chief characteristics.

In well-designed church bells a prominent note: the strike note, is heard. No partial corresponding to the pitch of that note is present in the unharmonic spectrum. An experimental rule, still adhered to by those workers not sufficiently misled by its theoretical improbability, asserts that the pitch of the strike note is an octave below that of the 5th partial of the bell spectrum. More physical or technical minded people, however, having found that the presence of the strike note is linked up with the presence of at least the 5th and the 7th partial, and having stated that the difference in frequency of those partials is sometimes almost equal to the pitch of the strike note, interpret it as a difference tone (formed within the ear) of those partials. In its essential features we are confronted with the "case of the missing fundamental" all over again.

On account of our hypothesis of the residue one might expect the following behaviour: if, in an unharmonic spectrum, by chance a number of partials are integer multiples of a certain frequency, an additional subjective component of corresponding pitch having a sharp tone quality and eventually great loudness should occur. If no such grouping is possible *no* extra component of definite pitch should be present; if several groupings are possible more than one additional component might occur. All of these expectations correspond to essential features of the strike note of bells.

We are thus led to suppose that the strike note of bells should, indeed, not be interpreted as a difference tone and moreover that it is another example of a residue. Conversely one might call the residue the "strike note" of periodic sounds.

The author wishes to thank Ir. R. VERMEULEN for his valuable criticism.

S. i. H., January 1940.

Mathematics. — *Ueber* BESSEL*sche*, STRUVE*sche und* LOMMEL*sche Funktionen.* (Zweite Mitteilung). Von C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT).

(Communicated at the meeting of February 24, 1940.)

§ 4. Aus (36), (55) und (56) geht hervor

$$K_\nu(2z) = \tfrac{1}{2} z^\nu\, G^{2,0}_{0,2}(z^2 \mid 0, -\nu) = \tfrac{1}{2} z^\nu\, G^{3,0}_{1,3}\left(z^2 \,\middle|\, \begin{matrix} \frac{1}{2} \\ \frac{1}{2}, 0, -\nu \end{matrix}\right). \quad . \quad (74)$$

Ebenso folgt aus (35), (55) und (56)

$$J_\nu(2z) = z^\nu\, G^{2,0}_{1,3}\left(z^2 \,\middle|\, \begin{matrix} \frac{1}{2} \\ \frac{1}{2}, 0, -\nu \end{matrix}\right). \quad . \quad . \quad . \quad . \quad . \quad (75)$$

und

$$J_\nu(2z) = z^{-\nu}\, G^{2,0}_{1,3}\left(z^2 \,\middle|\, \begin{matrix} \frac{1}{2} \\ \frac{1}{2}, \nu, 0 \end{matrix}\right). \quad . \quad . \quad . \quad . \quad . \quad (76)$$

Wendet man nun (26) auf die rechte Seite von (74) an, so erhält man

$$K_\nu(2z) = \frac{z^\nu}{\Gamma(\alpha - \frac{1}{2})} \int_1^\infty G^{3,0}_{1,3}\left(z^2 u^2 \,\middle|\, \begin{matrix} \frac{1}{2} \\ \alpha, 0, -\nu \end{matrix}\right) (u^2-1)^{\alpha - \frac{3}{2}}\, u^{1-2\alpha}\, du.$$

Nimmt man hierin $\alpha = \nu$, so bekommt man mit Rücksicht auf (49) [26])

$$K_\nu(2z) = \frac{2z^\nu}{\Gamma(\nu - \frac{1}{2}) \sqrt{\pi}} \int_1^\infty K_\nu^2(zu)\,(u^2-1)^{\nu-\frac{3}{2}}\, u^{1-2\nu}\, du; \quad . \quad . \quad (77)$$

diese Beziehung gilt für $|\arg z| < \frac{1}{2}\pi$ und $\Re(\nu) > \frac{1}{2}$.

Auf analoge Weise findet man mit Hilfe von (75) und (45)

$$J_\nu(2z) = -\frac{2z^\nu \sqrt{\pi}}{\Gamma(\nu - \frac{1}{2})} \int_1^\infty J_\nu(zu)\, Y_\nu(zu)\,(u^2-1)^{\nu-\frac{3}{2}}\, u^{1-2\nu}\, du; \quad . \quad (78)$$

hierin ist $z > 0$ und $\Re(\nu) > \frac{1}{2}$.

Aus (76) und (46) folgt schliesslich

$$J_\nu(2z)\sin\nu\pi = \frac{z^{-\nu}\sqrt{\pi}}{\Gamma(-\nu - \frac{1}{2})} \int_1^\infty \{J_{-\nu}^2(zu) - J_\nu^2(zu)\}\,(u^2-1)^{-\nu-\frac{3}{2}}\, u^{1+2\nu}\, du; \quad (79)$$

hierin ist $z > 0$ und $\Re(\nu) < -\frac{1}{2}$.

---

26) $G^{m,n}_{p,q}\left(\zeta \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$ ist eine symmetrische Funktion von $b_1, \ldots, b_m$.

§ 5. Für die Funktion $Y_\nu(z)$ gilt wegen (26) und (37)

$$Y_\nu(z) = \frac{2(-1)^h}{\Gamma(\alpha+\frac{1}{2}\nu)} \int_1^\infty G_{1,3}^{2,0}\left(\tfrac{1}{4} z^2 u^2 \,\middle|\, \begin{matrix} -\frac{1}{2}\nu - h - \frac{1}{2} \\ \alpha, \frac{1}{2}\nu, -\frac{1}{2}\nu - h - \frac{1}{2} \end{matrix}\right) (u^2-1)^{\alpha+\frac{1}{2}\nu-1} u^{1-2\alpha}\, du; \quad (80)$$

hierin ist $z > 0$, $\Re(\nu) < \frac{3}{2}$, $h = 0, \pm 1, \pm 2, \ldots$ und α beliebig mit $0 < \Re(\alpha + \frac{1}{2}\nu) < \frac{3}{2} - \Re(\nu)$.

Nun folgt aus (56), (55) und (35)

$$G_{1,3}^{2,0}\left(\tfrac{1}{4}\zeta^2 \,\middle|\, \begin{matrix} -\frac{1}{2}\nu - h - \frac{1}{2} \\ -\frac{1}{2}\nu - h - \frac{1}{2}, \frac{1}{2}\nu, -\frac{1}{2}\nu - h - \frac{1}{2} \end{matrix}\right)$$

$$= G_{0,2}^{1,0}\left(\tfrac{1}{4}\zeta^2 \,\middle|\, \tfrac{1}{2}\nu, -\tfrac{1}{2}\nu - h - \tfrac{1}{2}\right) = \left(\tfrac{1}{2}\zeta\right)^{-h-\frac{1}{2}} J_{\nu+h+\frac{1}{2}}(\zeta).$$

Der Spezialfall von (80) mit $\alpha = -\frac{1}{2}\nu - h - \frac{1}{2}$ liefert daher

$$Y_\nu(z) = \frac{(-1)^h 2^{h+\frac{3}{2}} z^{-h-\frac{1}{2}}}{\Gamma(-h-\frac{1}{2})} \int_1^\infty J_{\nu+h+\frac{1}{2}}(zu)(u^2-1)^{-h-\frac{3}{2}} u^{\nu+h+\frac{3}{2}}\, du; \quad (81)$$

in dieser Relation ist $z > 0$, $h = -1, -2, -3, \ldots$ und $\Re(\nu) < h + 2$.

Ist $l = 0, \pm 1, \pm 2, \ldots$, so hat man mit Rücksicht auf (55) und (37)

$$G_{1,3}^{2,0}\left(\tfrac{1}{4}\zeta^2 \,\middle|\, \begin{matrix} -\frac{1}{2}\nu - h - \frac{1}{2} \\ l - \frac{1}{2}\nu, \frac{1}{2}\nu, -\frac{1}{2}\nu - h - \frac{1}{2} \end{matrix}\right)$$

$$= \left(\tfrac{1}{2}\zeta\right)^l G_{1,3}^{2,0}\left(\tfrac{1}{4}\zeta^2 \,\middle|\, \begin{matrix} -\frac{1}{2}\nu - \frac{1}{2}l - h - \frac{1}{2} \\ \frac{1}{2}l - \frac{1}{2}\nu, \frac{1}{2}\nu - \frac{1}{2}l, -\frac{1}{2}\nu - \frac{1}{2}l - h - \frac{1}{2} \end{matrix}\right) = (-1)^{l+h} \left(\tfrac{1}{2}\zeta\right)^l Y_{\nu-l}(\zeta).$$

Für $\alpha = l - \frac{1}{2}\nu$ geht (80) also in

$$Y_\nu(z) = \frac{(-1)^l 2^{1-l} z^l}{\Gamma(l)} \int_1^\infty Y_{\nu-l}(zu)(u^2-1)^{l-1} u^{\nu-l+1}\, du \quad . \quad . \quad (82)$$

über; hierin ist $z > 0$, $l = 1, 2, 3, \ldots$ und $\Re(\nu) < \frac{3}{2} - l$.

Ist $z > 0$, $\Re(\nu) > -\frac{3}{2}$ und $0 < \Re(\alpha - \frac{1}{2}\nu) < \Re(\nu) + \frac{3}{2}$, so folgt aus (26) und (37)

$$Y_\nu(z) = \frac{2(-1)^h}{\Gamma(\alpha-\frac{1}{2}\nu)} \int_1^\infty G_{1,3}^{2,0}\left(\tfrac{1}{4} z^2 u^2 \,\middle|\, \begin{matrix} -\frac{1}{2}\nu - h - \frac{1}{2} \\ -\frac{1}{2}\nu, \alpha, -\frac{1}{2}\nu - h - \frac{1}{2} \end{matrix}\right) (u^2-1)^{\alpha-\frac{1}{2}\nu-1} u^{1-2\alpha}\, du. \quad (83)$$

Nun hat man wegen (55) und (37)

$$G_{1,3}^{2,0}\left(\tfrac{1}{4}\zeta^2 \,\middle|\, \begin{matrix} -\frac{1}{2}\nu - h - \frac{1}{2} \\ -\frac{1}{2}\nu, \frac{1}{2}\nu + \lambda, -\frac{1}{2}\nu - h - \frac{1}{2} \end{matrix}\right) = (-1)^h \left(\tfrac{1}{2}\zeta\right)^\lambda Y_{\nu+\lambda}(\zeta);$$

aus (83) mit $\alpha = \frac{1}{2}\nu + \lambda$ ergibt sich also

$$Y_\nu(z) = \frac{2^{1-\lambda} z^\lambda}{\Gamma(\lambda)} \int_1^\infty Y_{\nu+\lambda}(zu)\,(u^2-1)^{\lambda-1} u^{-\nu-\lambda+1}\,du. \quad . \quad . \quad . \qquad (84)$$

Diese Beziehung gilt für $z > 0$, $\Re(\nu) > -\frac{3}{2}$ und beliebige Werte von λ mit $0 < \Re(\lambda) < \Re(\nu) + \frac{3}{2}$.

Wegen

$$Y_{\frac{1}{2}}(w) = -\sqrt{\frac{2}{\pi w}}\cos w, \qquad Y_{-\frac{1}{2}}(w) = \sqrt{\frac{2}{\pi w}}\sin w$$

geht (84) für $\lambda = \frac{1}{2} - \nu$ bezw. $\lambda = -\frac{1}{2} - \nu$ in

$$Y_\nu(z) = -\frac{2^{1+\nu} z^{-\nu}}{\Gamma(\frac{1}{2}-\nu)\sqrt{\pi}} \int_1^\infty \frac{\cos(zu)\,du}{(u^2-1)^{\nu+\frac{1}{2}}} \qquad [z > 0;\ |\Re(\nu)| < \tfrac{1}{2}] \quad (85)$$

bezw.

$$Y_\nu(z) = \frac{2^{2+\nu} z^{-\nu-1}}{\Gamma(-\frac{1}{2}-\nu)\sqrt{\pi}} \int_1^\infty \frac{\sin(zu)\,u\,du}{(u^2-1)^{\nu+\frac{3}{2}}} \qquad [z > 0;\ -1 < \Re(\nu) < -\tfrac{1}{2}] \quad (86)$$

über.

Die den Relationen (81) bezw. (82) entsprechenden Integraldarstellungen der Funktion $J_\nu(z)$ sind

$$\left.\begin{aligned} J_\nu(z) &= \frac{(-1)^{h+1} 2^{h+\frac{3}{2}} z^{-h-\frac{1}{2}}}{\Gamma(-h-\frac{1}{2})} \int_1^\infty Y_{\nu+h+\frac{1}{2}}(zu)\,(u^2-1)^{-h-\frac{3}{2}} u^{\nu+h+\frac{3}{2}}\,du \\ &[z > 0;\ h = -1, -2, -3, \ldots;\ \Re(\nu) < h+2] \end{aligned}\right\} . \qquad (87)$$

bezw.

$$\left.\begin{aligned} J_\nu(z) &= \frac{(-1)^l 2^{1-l} z^l}{\Gamma(l)} \int_1^\infty J_{\nu-l}(zu)\,(u^2-1)^{l-1} u^{\nu-l+1}\,du \\ &[z > 0;\ l = 1, 2, 3, \ldots;\ \Re(\nu) < \tfrac{3}{2} - l]. \end{aligned}\right\} . \quad . \qquad (88)$$

Formel (87) kann mit Hilfe von [27])

$$J_\nu(z) = \frac{J_{-\nu}(z) + Y_\nu(z)\sin\nu\pi}{\cos\nu\pi}$$

aus (58) (mit $-\nu$ statt ν und $\lambda = -h - \frac{1}{2}$) und (81) abgeleitet werden; auf analoge Weise folgt (88) aus (58) und (82).

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27) Siehe (32).

Die Relationen (81), (82), (84), (85), (86), (87) und (88) waren bekannt [28]).

§ 6. Die Anwendung von (7) auf (42) liefert

$$\left.\begin{aligned} S_{\mu,\nu}(z) = {} & \frac{2^{\mu}}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(\alpha+\beta)} \int_1^{\infty} G_{1,3}^{3,1}\left(\tfrac{1}{4}z^2u^2 \,\middle|\, \begin{matrix} \frac{1}{2}+\frac{1}{2}\mu \\ \frac{1}{2}+\frac{1}{2}\mu, \alpha, \beta \end{matrix}\right) \\ & \times {}_2F_1(\alpha-\tfrac{1}{2}\nu, \alpha+\tfrac{1}{2}\nu; \alpha+\beta; 1-u^2)\,(u^2-1)^{\alpha+\beta-1}\,u^{1-2\beta}\,du. \end{aligned}\right\} \quad (89)$$

Nun folgt aus (55) und (42)

$$G_{1,3}^{3,1}\left(\tfrac{1}{4}\zeta^2 \,\middle|\, \begin{matrix} \frac{1}{2}+\frac{1}{2}\mu \\ \frac{1}{2}+\frac{1}{2}\mu, \frac{1}{2}\lambda+\frac{1}{2}\sigma, \frac{1}{2}\lambda-\frac{1}{2}\sigma \end{matrix}\right)$$
$$= (\tfrac{1}{2}\zeta)^{\lambda}\, 2^{1+\lambda-\mu}\, \Gamma(\tfrac{1}{2}-\tfrac{1}{2}\mu+\tfrac{1}{2}\lambda-\tfrac{1}{2}\sigma)\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\mu+\tfrac{1}{2}\lambda+\tfrac{1}{2}\sigma)\,S_{\mu-\lambda,\sigma}(\zeta).$$

Die Funktion $S_{\mu,\nu}(z)$ besitzt daher wegen (89) (mit $\alpha=\frac{1}{2}\lambda+\frac{1}{2}\sigma$ und $\beta=\frac{1}{2}\lambda-\frac{1}{2}\sigma$) die Integraldarstellung

$$\left.\begin{aligned} S_{\mu,\nu}(z) = {} & \frac{2z^{\lambda}\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\lambda-\frac{1}{2}\sigma)\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\lambda+\frac{1}{2}\sigma)}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(\lambda)} \int_1^{\infty} S_{\mu-\lambda,\sigma}(zu) \\ & \times {}_2F_1(\tfrac{1}{2}\lambda+\tfrac{1}{2}\sigma-\tfrac{1}{2}\nu, \tfrac{1}{2}\lambda+\tfrac{1}{2}\sigma+\tfrac{1}{2}\nu; \lambda; 1-u^2)\,(u^2-1)^{\lambda-1}\,u^{\sigma+1}\,du. \end{aligned}\right\} \quad (90)$$

Hierin wird $z\neq 0$, $|\arg z|<\pi$ und $\Re(\mu\pm\nu)<1$ vorausgesetzt; λ und σ sind beliebige Zahlen mit $\Re(\lambda)>0$ und $\mu-\lambda\pm\sigma\neq 1, 3, 5, \ldots$.

Für $\sigma=\nu-\lambda$ findet man eine den Formeln (58), (63) und (84) entsprechende Beziehung, nämlich

$$S_{\mu,\nu}(z) = \frac{2z^{\lambda}\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu+\lambda)}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu)\,\Gamma(\lambda)} \int_1^{\infty} S_{\mu-\lambda,\nu-\lambda}(zu)\,(u^2-1)^{\lambda-1}\,u^{\nu-\lambda+1}\,du; \quad (91)$$

diese Beziehung gilt für $|\arg z|<\pi$, $\Re(\mu+\nu)<1$ und beliebige Werte von λ mit $\Re(\lambda)>0$.

Man bekommt eine mit (69) verwandte Integraldarstellung für $S_{\mu,\nu}(z)$, wenn man $\lambda=1-\sigma$ und $u=\cosh t$ setzt in (90). Denn durch diese Substitution geht (90) wegen (68) in

$$\left.\begin{aligned} S_{\mu,\nu}(z) = {} & \frac{2z^{1-\sigma}\,\Gamma(1-\frac{1}{2}\mu-\sigma)\,\Gamma(1-\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu)} \\ & \times \int_0^{\infty} S_{\mu+\sigma-1,\sigma}(z\cosh t)\,P^{\sigma}_{\frac{1}{2}\nu-\frac{1}{2}}(\cosh 2t)\,\sinh^{1-\sigma}t\,\cosh t\,dt \end{aligned}\right\}. \quad (92)$$

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28) NIELSEN, [16], 222—223; WATSON, [17], § 6.13, Formel (4). Für (86) vergl. man MAYR, [5], 230, Formel (3a).

über; in dieser Relation ist $|\arg z| < \pi$, $\Re(\mu \pm \nu) < 1$ und σ beliebig mit $\Re(\sigma) < 1$ und $\frac{1}{2}\mu + \sigma \neq 1$.

Setzt man $\sigma = -\frac{1}{2}$ und $u = \cosh t$ in (90), so erhält man mit Rücksicht auf (73) (ich benutze $S_{\mu,\nu} = S_{\mu,-\nu}$)

$$\left.\begin{aligned} S_{\mu,\nu}(z) &= \frac{2^{\mu+\frac{1}{2}} z^{\lambda} \Gamma(\frac{1}{2}-\mu+\lambda)\sqrt{\pi}}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu)} \\ &\times \int_0^\infty S_{\mu-\lambda,\frac{1}{2}}(z\cosh t)\, P_{\nu-\frac{1}{2}}^{1-\lambda}(\cosh t) \sinh^{\lambda} t \cosh^{\frac{1}{2}} t\, dt; \end{aligned}\right\} \quad . \quad . \quad (93)$$

hierin ist $|\arg z| < \pi$, $\Re(\mu \pm \nu) < 1$ und λ beliebig mit $\Re(\lambda) > 0$ und $\mu - \lambda \neq \frac{1}{2}$.

Der Spezialfall von (92) mit $\sigma = \frac{1}{2}$ liefert wegen (70)

$$S_{\mu,\nu}(z) = \frac{2^{\mu+1} z^{\frac{1}{2}} \Gamma(1-\mu)}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu)} \int_0^\infty S_{\mu-\frac{1}{2},\frac{1}{2}}(z\cosh t)\cosh \nu t \cosh^{\frac{1}{2}} t\, dt;$$

diese Beziehung, gültig für $|\arg z| < \pi$ und $\Re(\mu \pm \nu) < 1$, kommt auch zum Vorschein, wenn man $\lambda = \frac{1}{2}$ setzt in (93).

§ 7. Aus (41) und (42) geht hervor

$$\mathbf{H}_\nu(z) - Y_\nu(z) = \frac{2^{1-\nu}}{\Gamma(\frac{1}{2}+\nu)\sqrt{\pi}} S_{\nu,\nu}(z). \quad . \quad . \quad . \quad . \quad (94)$$

Mit Hilfe dieser Beziehung kann man aus (90), (91), (92) und (93) Integraldarstellungen für $\mathbf{H}_\nu(z) - Y_\nu(z)$ ableiten. Aus (94) und (91) folgt z.B.

$$\left.\begin{aligned} \{\mathbf{H}_\nu(z) - Y_\nu(z)\}\cos(\nu-\lambda)\pi &= \frac{2^{1-\lambda} z^{\lambda} \cos \nu\pi}{\Gamma(\lambda)} \\ &\times \int_1^\infty \{\mathbf{H}_{\nu-\lambda}(zu) - Y_{\nu-\lambda}(zu)\}(u^2-1)^{\lambda-1} u^{\nu-\lambda+1}\, du; \end{aligned}\right\} \quad . \quad . \quad (95)$$

hierin ist $|\arg z| < \pi$, $\Re(\nu) < \frac{1}{2}$ und λ beliebig mit $\Re(\lambda) > 0$.

Ist $|\arg z| < \pi$ und $\Re(\alpha - \frac{1}{2}\nu) > \frac{1}{2}$, so findet man durch Anwendung von (26) auf (41)

$$\left.\begin{aligned} \mathbf{H}_\nu(2z) - Y_\nu(2z) &= \frac{2\cos \nu\pi}{\Gamma(\alpha-\frac{1}{2}\nu-\frac{1}{2})\pi^2} \\ &\times \int_1^\infty G_{1,3}^{3,1}\left(z^2u^2 \,\middle|\, \begin{matrix} \frac{1}{2}+\frac{1}{2}\nu \\ \alpha, -\frac{1}{2}\nu, \frac{1}{2}\nu \end{matrix}\right)(u^2-1)^{\alpha-\frac{1}{2}\nu-\frac{3}{2}} u^{1-2\alpha}\, du. \end{aligned}\right\} \quad . \quad (96)$$

Nun hat man wegen (55) und (50)

$$G_{1,3}^{3,1}\left(\zeta^2 \,\middle|\, \begin{matrix} \frac{1}{2}+\frac{1}{2}\nu \\ \frac{3}{2}\nu, -\frac{1}{2}\nu, \frac{1}{2}\nu \end{matrix}\right) = \zeta^\nu G_{1,3}^{3,1}\left(\zeta^2 \,\middle|\, \begin{matrix} \frac{1}{2} \\ \nu, -\nu, 0 \end{matrix}\right) = \frac{\zeta^\nu \pi^{\frac{5}{2}}}{2\cos\nu\pi} H_\nu^{(1)}(\zeta) H_\nu^{(2)}(\zeta);$$

aus (96) mit $\alpha = \frac{3}{2}\nu$ ergibt sich somit, falls $|\arg z| < \pi$ und $\Re(\nu) > \frac{1}{2}$ ist,

$$\mathbf{H}_\nu(2z) - Y_\nu(2z) = \frac{z^\nu \sqrt{\pi}}{\Gamma(\nu-\frac{1}{2})} \int_1^\infty H_\nu^{(1)}(zu) H_\nu^{(2)}(zu) (u^2-1)^{\nu-\frac{3}{2}} u^{1-2\nu}\, du. \quad (97)$$

Auf analoge Weise findet man mit Hilfe von (38) und (44)

$$\mathbf{H}_\nu(2z) = \frac{2z^\nu \sqrt{\pi}}{\Gamma(\nu-\frac{1}{2})} \int_1^\infty J_\nu^2(zu)(u^2-1)^{\nu-\frac{3}{2}} u^{1-2\nu}\, du; \quad . \quad . \quad . \quad (98)$$

hierin ist $z > 0$ und $\Re(\nu) > \frac{1}{2}$.

Ebenso folgt aus (39) und (48)

$$I_\nu(2z) - \mathbf{L}_\nu(2z) = \frac{4z^\nu}{\Gamma(\nu-\frac{1}{2})\sqrt{\pi}} \int_1^\infty I_\nu(zu) K_\nu(zu)(u^2-1)^{\nu-\frac{3}{2}} u^{1-2\nu}\, du; \quad . \quad (99)$$

diese Beziehung gilt für $|\arg z| < \frac{1}{2}\pi$ und $\Re(\nu) > \frac{1}{2}$.

Schliesslich folgt aus (40) und (47)

$$\{I_{-\nu}(2z) - \mathbf{L}_\nu(2z)\}\sin\nu\pi = \frac{z^\nu\sqrt{\pi}}{\Gamma(\nu-\frac{1}{2})} \int_1^\infty \{I_{-\nu}^2(zu) - I_\nu^2(zu)\}(u^2-1)^{\nu-\frac{3}{2}} u^{1-2\nu}\, du; (100)$$

hierin ist $|\arg z| < \frac{1}{2}\pi$ und $\Re(\nu) > \frac{1}{2}$.

Die Relationen (97), (98), (99) und (100) entsprechen (77), (78) und (79); Formel (98) war bekannt [29]).

Mit Hilfe von (26) kann man auch Umkehrformeln [30]) für (97), (98), (99) und (100) ableiten. Die Umkehrformel von (98) lautet z.B.

$$J_\nu^2(z) = \frac{2z^{-\nu}}{\Gamma(\frac{1}{2}-\nu)\sqrt{\pi}} \int_1^\infty \mathbf{H}_\nu(2zu)(u^2-1)^{-\nu-\frac{1}{2}} u^{-\nu}\, du;$$

hierin wird $z > 0$ und $|\Re(\nu)| < \frac{1}{2}$ vorausgesetzt. Diese Beziehung kommt zum Vorschein, wenn man (26) mit $\alpha = \frac{1}{2}$ auf (44) anwendet und (38) benutzt.

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29) WATSON, [17], 417, Formel (9).

30) Die Umkehrformel von (26) ist

$$G_{p,q}^{m,n}\left(\zeta \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ \alpha, b_2,\ldots,b_q \end{matrix}\right) = \frac{1}{\Gamma(b_1-\alpha)} \int_1^\infty G_{p,q}^{m,n}\left(\zeta v \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right)(v-1)^{b_1-\alpha-1} v^{-b_1}\, dv.$$

Aus (26) und (40) geht hervor

$$I_{-\nu}(z)-\mathbf{L}_\nu(z)=\frac{2\cos\nu\pi}{\Gamma(\alpha+\frac{1}{2}\nu)\pi}\int_1^\infty G_{1,3}^{2,1}\left(\tfrac{1}{4}z^2u^2\,\middle|\,\begin{matrix}\frac{1}{2}+\frac{1}{2}\nu\\ \frac{1}{2}+\frac{1}{2}\nu,\,\alpha,\,\frac{1}{2}\nu\end{matrix}\right)(u^2-1)^{\alpha+\frac{1}{2}\nu-1}\,u^{1-2\alpha}\,du;$$

ersetzt man nun α durch $\lambda-\frac{1}{2}\nu$, so erhält man wegen (40) die mit (58), (63), (84) und (95) verwandte Formel

$$\{I_{-\nu}(z)-\mathbf{L}_\nu(z)\}\cos(\nu-\lambda)\pi=\frac{2^{1-\lambda}z^\lambda\cos\nu\pi}{\Gamma(\lambda)}\int_1^\infty\{I_{-\nu+\lambda}(zu)-\mathbf{L}_{\nu-\lambda}(zu)\}(u^2-1)^{\lambda-1}\,u^{\nu-\lambda+1}\,du;$$

hierin ist $|\arg z|<\frac{1}{2}\pi$, $\Re(\nu)<\frac{1}{2}$ und λ beliebig mit $\Re(\lambda)>0$.

Die entsprechenden Integraldarstellungen der Funktionen $\mathbf{H}_\nu(z)$ und $I_\nu(z)-\mathbf{L}_\nu(z)$ sind

$$\mathbf{H}_\nu(z)=\frac{2^{1-\lambda}z^\lambda}{\Gamma(\lambda)}\int_0^1\mathbf{H}_{\nu-\lambda}(zu)\,(1-u^2)^{\lambda-1}\,u^{\nu-\lambda+1}\,du \quad . \; . \quad (101)$$

(wo $\Re(\nu)>-\frac{3}{2}$ und λ beliebig mit $0<\Re(\lambda)<\Re(\nu)+\frac{3}{2}$) und

$$I_\nu(z)-\mathbf{L}_\nu(z)=\frac{2^{1-\lambda}z^\lambda}{\Gamma(\lambda)}\int_0^1\{I_{\nu-\lambda}(zu)-\mathbf{L}_{\nu-\lambda}(zu)\}(1-u^2)^{\lambda-1}\,u^{\nu-\lambda+1}\,du \quad (102)$$

(wo $\Re(\nu)>-1$ und λ beliebig mit $0<\Re(\lambda)<\Re(\nu)+1$).

Der Beweis von (101) bezw. (102) geht durch Entwicklung der Funktionen $\mathbf{H}_{\nu-\lambda}(zu)$ bezw. $I_{\nu-\lambda}(zu)-\mathbf{L}_{\nu-\lambda}(zu)$ (siehe (31) und (34)) und gliedweise Integration [31]).

Nun folgt aus (34) und (31)

$$\mathbf{H}_{-\frac{1}{2}}(w)=\sqrt{\frac{2}{\pi w}}\sin w, \qquad I_{-\frac{1}{2}}(w)-\mathbf{L}_{-\frac{1}{2}}(w)=\sqrt{\frac{2}{\pi w}}\,e^{-w};$$

die Spezialfälle mit $\lambda=\nu+\frac{1}{2}$ von (101) bezw. (102) liefern also [32])

$$\mathbf{H}_\nu(z)=\frac{2^{1-\nu}z^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}\int_0^1(1-u^2)^{\nu-\frac{1}{2}}\sin(zu)\,du$$

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[31]) Die (101) und (102) entsprechende Integraldarstellung für $J_\nu(z)$, nämlich

$$J_\nu(z)=\frac{2^{1-\lambda}z^\lambda}{\Gamma(\lambda)}\int_0^1 J_{\nu-\lambda}(zu)\,(1-u^2)^{\lambda-1}\,u^{\nu-\lambda+1}\,du,$$

ist schon lange bekannt; man vergl. WATSON, [17], 373, Formel (1).

[32]) Ich nehme an, dass $\Re(\nu)>-\frac{1}{2}$ ist.

bezw.

$$I_\nu(z) - \mathbf{L}_\nu(z) = \frac{2^{1-\nu} z^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \int_0^1 (1-u^2)^{\nu-\frac{1}{2}} e^{-zu}\, du.$$

Diese Beziehungen waren bekannt [33]).

§ 8. Durch Anwendung von (7) auf (49) erhält man

$$\left.\begin{aligned} K_\nu^2(z) = \frac{\sqrt{\pi}}{\Gamma(\alpha+\beta)} \int_1^\infty G_{1,3}^{3,0}\left(z^2 u^2 \,\middle|\, \begin{matrix} \frac{1}{2} \\ \alpha, \beta, 0 \end{matrix}\right) \\ \times {}_2F_1(\alpha-\nu, \alpha+\nu; \alpha+\beta; 1-u^2)(u^2-1)^{\alpha+\beta-1} u^{1-2\beta}\, du. \end{aligned}\right\} . \tag{103}$$

Nun folgt aus (56), (55) und (36)

$$G_{1,3}^{3,0}\left(\zeta^2 \,\middle|\, \begin{matrix} \frac{1}{2} \\ \frac{1}{2}, \beta, 0 \end{matrix}\right) = G_{0,2}^{2,0}(\zeta^2 \,|\, \beta, 0) = 2\zeta^\beta K_\beta(2\zeta).$$

Setzt man $\alpha = \frac{1}{2}$ und $u = \cosh t$ in (103), so findet man also mit Rücksicht auf (68)

$$K_\nu^2(z) = 2z^\beta \sqrt{\pi} \int_0^\infty K_\beta(2z\cosh t)\, P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\beta}(\cosh 2t) \sinh^{\beta+\frac{1}{2}} t \cosh^{\frac{1}{2}} t\, dt; \tag{104}$$

hierin ist $|\arg z| < \frac{1}{2}\pi$ und β beliebig mit $\Re(\beta) > -\frac{1}{2}$.

Mit Hilfe von (46) und (35) beweist man ganz analog

$$J_{-\nu}^2(z) - J_\nu^2(z) = \frac{4z^\beta \sin\nu\pi}{\sqrt{\pi}} \int_0^\infty J_\beta(2z\cosh t)\, P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\beta}(\cosh 2t) \sinh^{\beta+\frac{1}{2}} t \cosh^{\frac{1}{2}} t\, dt;$$

diese Beziehung gilt für $z > 0$, $|\Re(\nu)| < 1$ und beliebige Werte von β mit $-\frac{1}{2} < \Re(\beta) < -2|\Re(\nu)| + \frac{3}{2}$.

Die Anwendung von (7) auf (50) ergibt

$$\left.\begin{aligned} H_\nu^{(1)}(z) H_\nu^{(2)}(z) = \frac{4\cos\nu\pi}{\Gamma(\alpha+\beta)\pi^{\frac{5}{2}}} \int_1^\infty G_{1,3}^{3,1}\left(z^2 u^2 \,\middle|\, \begin{matrix} \frac{1}{2} \\ \alpha, \beta, 0 \end{matrix}\right) \\ \times {}_2F_1(\alpha-\nu, \alpha+\nu; \alpha+\beta; 1-u^2)(u^2-1)^{\alpha+\beta-1} u^{1-2\beta}\, du. \end{aligned}\right\} . \tag{105}$$

Nun hat man wegen (55) und (41)

$$\begin{aligned} G_{1,3}^{3,1}\left(\zeta^2 \,\middle|\, \begin{matrix} \frac{1}{2} \\ \frac{1}{2}, \beta, 0 \end{matrix}\right) &= \zeta^\beta G_{1,3}^{3,1}\left(\zeta^2 \,\middle|\, \begin{matrix} \frac{1}{2} - \frac{1}{2}\beta \\ \frac{1}{2} - \frac{1}{2}\beta, \frac{1}{2}\beta, -\frac{1}{2}\beta \end{matrix}\right) \\ &= \frac{\zeta^\beta \pi^2}{\cos\beta\pi} \{\mathbf{H}_{-\beta}(2\zeta) - Y_{-\beta}(2\zeta)\}. \end{aligned}$$

---

33) Man vergl. WATSON, [17], 330, Formeln (1) und (2).

Aus (105) mit $\alpha = \frac{1}{2}$ und $u = \cosh t$ folgt also mit Rücksicht auf (68)

$$H_\nu^{(1)}(z)\, H_\nu^{(2)}(z) \cos\beta\pi = \frac{4 z^\beta \cos\nu\pi}{\sqrt{\pi}} \int_0^\infty \{\mathbf{H}_{-\beta}(2z\cosh t) - Y_{-\beta}(2z\cosh t)\}$$

$$\times P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\beta}(\cosh 2t) \sinh^{\beta+\frac{1}{2}} t \cosh^{\frac{1}{2}} t\, dt;$$

in dieser Beziehung ist $|\arg z| < \pi$, $|\Re(\nu)| < \frac{1}{2}$ und β beliebig mit $\Re(\beta) > -\frac{1}{2}$.

In ähnlicher Weise findet man mit Hilfe von (47) und (40)

$$\{I_{-\nu}^2(z) - I_\nu^2(z)\} \cos\beta\pi = \frac{2 z^\beta \sin 2\nu\pi}{\sqrt{\pi}} \int_0^\infty \{I_\beta(2z\cosh t) - \mathbf{L}_{-\beta}(2z\cosh t)\}$$

$$\times P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\beta}(\cosh 2t) \sinh^{\beta+\frac{1}{2}} t \cosh^{\frac{1}{2}} t\, dt;$$

in dieser Beziehung wird $|\arg z| < \frac{1}{2}\pi$, $|\Re(\nu)| < \frac{1}{2}$ und $\Re(\beta) > -\frac{1}{2}$ vorausgesetzt.

Eine verwandte Integraldarstellung für $J_\nu(z)\, J_{-\nu}(z)$ ist

$$J_\nu(z)\, J_{-\nu}(z) = \frac{2 z^\beta}{\sqrt{\pi}} \int_0^{\frac{1}{2}\pi} J_{-\beta}(2z\cos\varphi)\, \mathbf{P}_{\nu-\frac{1}{2}}^{\frac{1}{2}-\beta}(\cos 2\varphi) \sin^{\beta+\frac{1}{2}}\varphi \cos^{\frac{1}{2}}\varphi\, d\varphi; \quad (106)$$

hierin ist β beliebig mit $-\frac{1}{2} < \Re(\beta) < 1$.

Denn die zugeordnete LEGENDREsche Funktion erster Art wird für $-1 < w < 1$ definiert durch [34])

$$\mathbf{P}_n^m(w) = \frac{(1+w)^{\frac{1}{2}m}(1-w)^{-\frac{1}{2}m}}{\Gamma(1-m)}\ {}_2F_1(-n, 1+n; 1-m; \tfrac{1}{2} - \tfrac{1}{2}w).$$

Die rechte Seite von (106) ist also mit Rücksicht auf (31) gleich [35])

$$\frac{1}{\Gamma(1-\beta)\,\Gamma(\frac{1}{2}+\beta)\sqrt{\pi}} \int_0^1 {}_0F_1(1-\beta; -z^2 u) \,.\, {}_2F_1(\tfrac{1}{2}-\nu, \tfrac{1}{2}+\nu; \tfrac{1}{2}+\beta; 1-u)\,(1-u)^{\beta-\frac{1}{2}} u^{-\beta}\, du.$$

Ist $-\frac{1}{2} < \Re(\beta) < 1$, so besitzt dieser Ausdruck den Wert [36])

$$\frac{1}{\Gamma(1-\nu)\,\Gamma(1+\nu)}\ {}_1F_2(\tfrac{1}{2}; 1-\nu, 1+\nu; -z^2),$$

und diese Funktion ist gleich [37]) $J_\nu(z)\, J_{-\nu}(z)$, so dass Formel (106) bewiesen ist.

---

34) Man vergl. HOBSON [2], 227.

35) Ich ersetze $\cos^2\varphi$ durch u.

36) Siehe [13], Formel (16).

37) WATSON, [17], 147, Formel (7).

§ 9. Für das Produkt $K_\mu(z)\,K_\nu(z)$ gilt wegen (7) und (52)

$$\left.\begin{aligned} K_\mu(z)\,K_\nu(z) = \frac{\sqrt{\pi}}{\Gamma(\alpha+\beta)} \int_1^\infty G_{2,4}^{4,0}\left(z^2u^2 \,\middle|\, \begin{matrix} 0,\ \frac{1}{2} \\ \alpha,\beta,\frac{1}{2}\mu+\frac{1}{2}\nu,-\frac{1}{2}\mu-\frac{1}{2}\nu \end{matrix}\right) \\ \times\, {}_2F_1\left(\alpha-\tfrac{1}{2}\mu+\tfrac{1}{2}\nu,\ \alpha+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu;\ \alpha+\beta;\ 1-u^2\right)(u^2-1)^{\alpha+\beta-1}\,u^{1-2\beta}\,du. \end{aligned}\right\} \quad (107)$$

Nun folgt aus (56) und (36)

$$\begin{aligned} &G_{2,4}^{4,0}\left(\zeta^2 \,\middle|\, \begin{matrix} 0,\ \frac{1}{2} \\ 0,\frac{1}{2},\frac{1}{2}\mu+\frac{1}{2}\nu,-\frac{1}{2}\mu-\frac{1}{2}\nu \end{matrix}\right) \\ &= G_{0,2}^{2,0}\left(\zeta^2 \,\middle|\, \tfrac{1}{2}\mu+\tfrac{1}{2}\nu,-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu\right) = 2\,K_{\mu+\nu}(2\zeta); \end{aligned}$$

weiter hat man [38])

$$ {}_2F_1\left(-\lambda,\lambda;\tfrac{1}{2};-\sinh^2 t\right) = \cosh 2\lambda t. \quad . \quad . \quad . \quad . \quad (108)$$

Aus (107) mit $\alpha = 0$, $\beta = \frac{1}{2}$ und $u = \cosh t$ ergibt sich also

$$K_\mu(z)\,K_\nu(z) = 2\int_0^\infty K_{\mu+\nu}(2z\cosh t)\cosh(\mu-\nu)\,t\,dt; \quad . \quad . \quad (109)$$

diese Beziehung, gültig für $|\arg z| < \frac{1}{2}\pi$, war schon lange bekannt [39]).

Durch analoge Anwendung von (7) auf (51) bekommt man

$$T_{\nu,\mu}(z) = \frac{8}{\pi i}\int_0^\infty J_{\mu+\nu}(2z\cosh t)\cosh(\mu-\nu)\,t\,dt; \quad . \quad . \quad . \quad (110)$$

hierin ist $z > 0$ und $|\Re(\mu-\nu)| < \frac{3}{2}$.

Mit Hilfe von (7) und (54) findet man für die Funktion $V_{\nu,\mu}(z)$

$$\left.\begin{aligned} V_{\nu,\mu}(z) = \frac{4\cos(\frac{1}{2}\mu+\frac{1}{2}\nu)\pi\cos(\frac{1}{2}\mu-\frac{1}{2}\nu)\pi}{\Gamma(\alpha+\beta)\,\pi^{\frac{5}{2}}} \int_1^\infty G_{2,4}^{4,1}\left(z^2u^2 \,\middle|\, \begin{matrix} \frac{1}{2},\ 0 \\ \alpha,\beta,\frac{1}{2}\mu+\frac{1}{2}\nu,-\frac{1}{2}\mu-\frac{1}{2}\nu \end{matrix}\right) \\ \times\, {}_2F_1\left(\alpha-\tfrac{1}{2}\mu+\tfrac{1}{2}\nu,\ \alpha+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu;\ \alpha+\beta;\ 1-u^2\right)(u^2-1)^{\alpha+\beta-1}\,u^{1-2\beta}\,du. \end{aligned}\right\} \quad (111)$$

Nun hat man wegen (56) und (42)

$$\begin{aligned} G_{2,4}^{4,1}\left(\zeta^2 \,\middle|\, \begin{matrix} \frac{1}{2},\ 0 \\ 0,\frac{1}{2},\frac{1}{2}\mu+\frac{1}{2}\nu,-\frac{1}{2}\mu-\frac{1}{2}\nu \end{matrix}\right) &= G_{1,3}^{3,1}\left(\zeta^2 \,\middle|\, \begin{matrix} \frac{1}{2} \\ \frac{1}{2},\frac{1}{2}\mu+\frac{1}{2}\nu,-\frac{1}{2}\mu-\frac{1}{2}\nu \end{matrix}\right) \\ &= 2\,\Gamma\left(\tfrac{1}{2}-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu\right)\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu\right)S_{0,\mu+\nu}(2\zeta). \end{aligned}$$

---

38) Man vergl. [6], Formel (42).

39) WATSON, [17], 440. Siehe auch MEIJER, [8], 16—17; [9], 483.

Setzt man $\alpha=0$, $\beta=\frac{1}{2}$ und $u=\cosh t$ in (111), so erhält man also mit Rücksicht auf (108)

$$V_{\nu,\mu}(z)=\frac{8\cos(\frac{1}{2}\mu-\frac{1}{2}\nu)\pi}{\pi^2}\int_0^\infty S_{0,\mu+\nu}(2z\cosh t)\cosh(\mu-\nu)t\,dt.$$

Diese Beziehung, gültig für $|\arg z|<\pi$ und $|\Re(\mu-\nu)|<1$, ist mit (109) und (110) verwandt.

Durch Anwendung von (7) auf (53) findet man die entsprechende Integraldarstellung der Funktion $U_{\nu,\mu}(z)$, nämlich

$$U_{\nu,\mu}(z)=\frac{8(\mu+\nu)\sin(\frac{1}{2}\nu-\frac{1}{2}\mu)\pi}{\pi^2}\int_0^\infty S_{-1,\mu+\nu}(2z\cosh t)\cosh(\mu-\nu)t\,dt;$$

hierin ist $|\arg z|<\pi$ und $|\Re(\mu-\nu)|<2$.

§ 10. Die Anwendung von (7) auf (54) liefert nicht nur (111), sondern auch

$$\left.\begin{aligned}V_{\nu,\mu}(z)=&\frac{4\cos(\frac{1}{2}\mu+\frac{1}{2}\nu)\pi\cos(\frac{1}{2}\mu-\frac{1}{2}\nu)\pi}{\Gamma(\alpha+\beta-\mu)\pi^{\frac{5}{2}}}\int_1^\infty G_{2,4}^{4,1}\left(z^2u^2\,\middle|\,\begin{matrix}\frac{1}{2},0\\ \alpha,\beta,\frac{1}{2}\nu-\frac{1}{2}\mu,-\frac{1}{2}\mu-\frac{1}{2}\nu\end{matrix}\right)\\ &\times{}_2F_1(\alpha-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu,\alpha-\tfrac{1}{2}\mu+\tfrac{1}{2}\nu;\alpha+\beta-\mu;1-u^2)(u^2-1)^{\alpha+\beta-\mu-1}u^{1-2\beta}\,du.\end{aligned}\right\}\tag{112}$$

Nun gilt wegen (56), (55) und (42)

$$G_{2,4}^{4,1}\left(\zeta^2\,\middle|\,\begin{matrix}\frac{1}{2},0\\ 0,\frac{1}{2},\frac{1}{2}\nu-\frac{1}{2}\mu,-\frac{1}{2}\mu-\frac{1}{2}\nu\end{matrix}\right)=G_{1,3}^{3,1}\left(\zeta^2\,\middle|\,\begin{matrix}\frac{1}{2}\\ \frac{1}{2},\frac{1}{2}\nu-\frac{1}{2}\mu,-\frac{1}{2}\mu-\frac{1}{2}\nu\end{matrix}\right)$$

$$=\zeta^{-\mu}G_{1,3}^{3,1}\left(\zeta^2\,\middle|\,\begin{matrix}\frac{1}{2}+\frac{1}{2}\mu\\ \frac{1}{2}+\frac{1}{2}\mu,\frac{1}{2}\nu,-\frac{1}{2}\nu\end{matrix}\right)=2^{1-\mu}\zeta^{-\mu}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\mu+\tfrac{1}{2}\nu)S_{\mu,\nu}(2\zeta).$$

Aus (112) mit $\alpha=0$, $\beta=\frac{1}{2}$ und $u=\cosh t$ ergibt sich also mit Rücksicht auf (73)

$$V_{\nu,\mu}(z)=\frac{2^{-2\mu+\frac{5}{2}}z^{-\mu}}{\Gamma(\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu)\Gamma(\frac{1}{2}+\frac{1}{2}\mu-\frac{1}{2}\nu)\sqrt{\pi}}$$

$$\times\int_0^\infty S_{\mu,\nu}(2z\cosh t)P_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(\cosh t)\sinh^{\frac{1}{2}-\mu}t\cosh^{-\mu}t\,dt;$$

hierin ist $|\arg z|<\pi$, $\Re(\mu)<\frac{1}{2}$ und $\Re(\mu\pm\nu)>-1$.

Auf analoge Weise findet man mit Hilfe von (53) und (42)

$$U_{\nu,\mu}(z) = -\frac{2^{-2\mu+\frac{7}{2}} z^{-\mu}}{\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}\mu-\frac{1}{2}\nu)\sqrt{\pi}}$$

$$\times \int_0^\infty S_{\mu-1,\nu}(2z\cosh t)\, P_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(\cosh t)\sinh^{\frac{1}{2}-\mu} t \cosh^{-\mu} t\, dt;$$

diese Beziehung gilt für $|\arg z| < \pi$, $\Re(\mu) < \frac{1}{2}$ und $\Re(\mu \pm \nu) > -2$.

Die (112) entsprechende Integraldarstellung der Funktion $K_\mu(z)K_\nu(z)$ ist wegen (7) und (52)

$$\left.\begin{aligned} K_\mu(z)K_\nu(z) = \frac{\sqrt{\pi}}{\Gamma(\alpha+\beta-\mu)}\int_1^\infty G_{2,4}^{4,0}\left(z^2u^2 \,\middle|\, \begin{matrix} 0,\ \frac{1}{2} \\ \alpha,\ \beta,\ \frac{1}{2}\nu-\frac{1}{2}\mu,\ -\frac{1}{2}\mu-\frac{1}{2}\nu \end{matrix}\right) \\ \times {}_2F_1(\alpha-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu, \alpha-\tfrac{1}{2}\mu+\tfrac{1}{2}\nu; \alpha+\beta-\mu; 1-u^2)(u^2-1)^{\alpha+\beta-\mu-1}u^{1-2\beta}\,du. \end{aligned}\right\} \quad (113)$$

Nun hat man infolge (56), (55) und (36)

$$G_{2,4}^{4,0}\left(\zeta^2 \,\middle|\, \begin{matrix} 0,\ \frac{1}{2} \\ 0,\ \frac{1}{2},\ \frac{1}{2}\nu-\frac{1}{2}\mu,\ -\frac{1}{2}\mu-\frac{1}{2}\nu \end{matrix}\right) = G_{0,2}^{2,0}(\zeta^2 \,|\, \tfrac{1}{2}\nu-\tfrac{1}{2}\mu, -\tfrac{1}{2}\mu-\tfrac{1}{2}\nu)$$

$$= \zeta^{-\mu} G_{0,2}^{2,0}(\zeta^2 \,|\, \tfrac{1}{2}\nu, -\tfrac{1}{2}\nu) = 2\zeta^{-\mu} K_\nu(2\zeta).$$

Aus (113) mit $\alpha = 0$, $\beta = \frac{1}{2}$ und $u = \cosh t$ geht also hervor mit Rücksicht auf (73)

$$K_\mu(z)K_\nu(z) = 2^{\frac{1}{2}-\mu} z^{-\mu}\sqrt{\pi}\int_0^\infty K_\nu(2z\cosh t)\, P_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(\cosh t)\sinh^{\frac{1}{2}-\mu} t\cosh^{-\mu} t\, dt;$$

diese Beziehung, gültig für $|\arg z| < \frac{1}{2}\pi$ und $\Re(\mu) < \frac{1}{2}$, habe ich früher[40]) auf andre Weise abgeleitet.

Ebenso beweist man, ausgehend von (51),

$$T_{\nu,\mu}(z) = \frac{2^{\frac{1}{2}-\mu} z^{-\mu}}{i\sqrt{\pi}}\int_0^\infty J_\nu(2z\cosh t)\, P_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(\cosh t)\sinh^{\frac{1}{2}-\mu} t\cosh^{-\mu} t\, dt;$$

hierin ist $z > 0$, $\Re(\mu) < \frac{1}{2}$ und $\Re(2\mu \pm \nu) > -\frac{3}{2}$.

Setzt man $\alpha = \frac{1}{2} - \frac{1}{2}\mu + \frac{1}{2}\nu$, $\beta = \frac{1}{2} - \frac{1}{2}\mu - \frac{1}{2}\nu$ und $u = \operatorname{cotgh} t$ in (113), so erhält man

$$\left.\begin{aligned} K_\mu(z)K_\nu(z) = \frac{\sqrt{\pi}}{\Gamma(1-2\mu)}\int_0^\infty G_{2,4}^{4,0}\left(z^2\operatorname{cotgh}^2 t \,\middle|\, \begin{matrix} 0,\ \frac{1}{2} \\ \frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu,\ \frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu,\ \frac{1}{2}\nu-\frac{1}{2}\mu,\ -\frac{1}{2}\mu-\frac{1}{2}\nu \end{matrix}\right) \\ \times {}_2F_1(\tfrac{1}{2}-\mu, \tfrac{1}{2}-\mu+\nu; 1-2\mu; 1-\operatorname{cotgh}^2 t)\sinh^{3\mu-\nu-2} t\cosh^{\mu+\nu} t\, dt. \end{aligned}\right\} \quad (114)$$

---

40) MEIJER, [9], 482, Formel (51).

Nun ist [41])

$$
\begin{aligned}
&{}_2F_1\left(\tfrac{1}{2}-m, 1-m+n; 1-2m; 1-\mathrm{tgh}^2 t\right) \\
&= 2^{-2m}\,\Gamma(1-m)\sinh^{-n-1} t \cosh^{n-2m+1} t\, P_n^m(\operatorname{cotgh} 2t);
\end{aligned}
$$

diese Beziehung geht mit Hilfe von (13) in

$$
\begin{aligned}
&{}_2F_1\left(\tfrac{1}{2}-m, 1-m+n; 1-2m; 1-\operatorname{cotgh}^2 t\right) \\
&= 2^{-2m}\,\Gamma(1-m)\sinh^{n-2m+1} t \cosh^{-n-1} t\, P_n^m(\operatorname{cotgh} 2t)
\end{aligned}
$$

über.

Für die in (114) vorkommende Funktion ${}_2F_1$ gilt daher

$$
\left.
\begin{aligned}
&{}_2F_1\left(\tfrac{1}{2}-\mu, \tfrac{1}{2}-\mu+\nu; 1-2\mu; 1-\operatorname{cotgh}^2 t\right) \\
&= 2^{-2\mu}\,\Gamma(1-\mu)\sinh^{\nu-2\mu+\frac{1}{2}} t \cosh^{-\nu-\frac{1}{2}} t\, P_{\nu-\frac{1}{2}}^{\mu}(\operatorname{cotgh} 2t).
\end{aligned}
\right\} \quad . \quad . \quad (115)
$$

Ferner hat man wegen (55) und (43)

$$
\left.
\begin{aligned}
&G_{2,4}^{4,0}\left(\zeta^2 \,\middle|\, \begin{matrix} 0, \tfrac{1}{2} \\ \tfrac{1}{2}-\tfrac{1}{2}\mu+\tfrac{1}{2}\nu, \tfrac{1}{2}-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu, \tfrac{1}{2}\nu-\tfrac{1}{2}\mu, -\tfrac{1}{2}\mu-\tfrac{1}{2}\nu \end{matrix}\right) \\
&= \zeta^{-\mu}\, G_{2,4}^{4,0}\left(\zeta^2 \,\middle|\, \begin{matrix} \tfrac{1}{2}\mu, \tfrac{1}{2}+\tfrac{1}{2}\mu \\ \tfrac{1}{2}+\tfrac{1}{2}\nu, \tfrac{1}{2}-\tfrac{1}{2}\nu, \tfrac{1}{2}\nu, -\tfrac{1}{2}\nu \end{matrix}\right) = 2^{\mu-\frac{1}{2}}\,\zeta^{-\mu-\frac{1}{2}}\sqrt{\pi}\, e^{-\zeta}\, W_{\frac{1}{2}-\mu,\nu}(2\zeta).
\end{aligned}
\right\} \quad (116)
$$

Die Funktion $K_\mu(z)\,K_\nu(z)$ besitzt also mit Rücksicht auf (114), (116) und (115) die Integraldarstellung

$$
\begin{aligned}
K_\mu(z)\,K_\nu(z) &= \frac{2^{\mu-\frac{1}{2}}\, z^{-\mu-\frac{1}{2}}\, \pi^{\frac{3}{2}}}{\Gamma\left(\frac{1}{2}-\mu\right)} \int_0^\infty e^{-z \operatorname{cotgh} t} \\
&\times W_{\frac{1}{2}-\mu,\nu}(2z \operatorname{cotgh} t)\, P_{\nu-\frac{1}{2}}^{\mu}(\operatorname{cotgh} 2t) \sinh^{2\mu-1} t \cosh^{-1} t\, dt;
\end{aligned}
$$

hierin ist $|\arg z| < \frac{1}{2}\pi$ und $\Re(\mu) < \frac{1}{2}$.

Genau so findet man, wenn man $\alpha = -\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu$, $\beta = -\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu$ und $u = \operatorname{cotgh} t$ setzt in (113),

$$
\begin{aligned}
K_\mu(z)\,K_\nu(z) &= \frac{2^{\mu+\frac{1}{2}}\, z^{-\mu-\frac{3}{2}}\, \pi^{\frac{3}{2}}}{\Gamma\left(-\frac{1}{2}-\mu\right)} \int_0^\infty e^{-z \operatorname{cotgh} t} \\
&\times W_{-\frac{1}{2}-\mu,\nu}(2z \operatorname{cotgh} t)\, P_{\nu-\frac{1}{2}}^{\mu+1}(\operatorname{cotgh} 2t) \sinh^{2\mu} t\, dt;
\end{aligned}
$$

diese Beziehung gilt für $|\arg z| < \frac{1}{2}\pi$ und $\Re(\mu) < -\frac{1}{2}$.

---

41) Man vergl. HOBSON, [2], 236, Formel (74).

Mathematics. — *Développements en série de polynomes d'*HERMITE *et de* LAGUERRE *à l'aide des transformations de* GAUSS *et de* HANKEL. III [26]). Par ERVIN FELDHEIM. (Communicated by Prof. J. G. VAN DER CORPUT).

(Communicated at the meeting of February 24, 1940.)

§ 5. *Quelques généralisations ultérieures.*

Cette partie du travail contient des généralisations pour certains résultats de I et II, et surtout pour la formule (21) qui a été établie dans un travail actuellement sous presse [27]). Nous allons montrer aussi que les développements (18), et une inversion de (60), permettent de donner une nouvelle interprétation pour quelques résultats connus.

1°. *Généralisation de l'équation intégrale* (21). Si nous remplaçons, dans la formule (46), y et z respectivement par $u+v$ et $u-v$, cette relation nous fournit la transformée de GAUSS $G_x(a, u)$ du produit $H_m\left(\frac{x+v}{\lambda}\right) H_n\left(\frac{x-v}{\mu}\right)$, u et v étant des valeurs quelconques de l'intervalle $(-\infty, +\infty)$, les autres paramètres restant assujettis aux conditions imposées pour (46). Si nous posons, dans la relation obtenue, $a=\lambda=\mu=1$, nous serons conduits à l'équation intégrale

$$\left.\begin{aligned} &\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(x-iu)^2} H_m(x+v)\, H_n(x-v)\, dx = \\ &\qquad = 2^m\, n!\, (v+iu)^{m-n}\, L_n^{(m-n)}(2u^2+2v^2) \qquad (m \geqq n) \end{aligned}\right\} \quad . \quad . \quad (61)$$

Cette équation résulte immédiatement de (20), et ne peut pas être considérée par suite comme une généralisation de celle-ci. Mais son inversion:

$$\left.\begin{aligned} &2^m\, n!\, \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(u+ix)^2} (v+iu)^{m-n}\, L_n^{(m-n)}(2u^2+2v^2)\, du = \\ &\qquad = H_m(x+v)\, H_n(x-v) \qquad (m \geqq n) \end{aligned}\right\} \quad . \quad . \quad (62)$$

---

26) Les parties I et II ont été publiées dans les Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, 224—239; 240—248 (1940).

27) FELDHEIM, [8]. Les nombres entre crochets renvoient à la Bibliographie placée à la fin de la 2e communication.

constitue bien une généralisation de la relation (21), qui ne peut pas être déduite du développement (16). Nous allons donner un peu plus tard le développement équivalent à (62), mais démontrons d'abord cette équation intégrale à l'aide de (61). On en tire

$$2^m n! \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(u+ix)^2} (v+iu)^{m-n} L_n^{(m-n)} (2u^2+2v^2)\, du =$$

$$= \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{-(u+ix)^2} du \int_{-\infty}^{+\infty} e^{-(y-iu)^2} H_m(y+v) H_n(y-v)\, dy.$$

Si l'intégration par rapport à u est d'abord effectuée entre les limites $-\frac{U}{2}$ et $\frac{U}{2}$, le produit des deux intégrales précédentes pourra être considéré comme une intégrale double qui est sûrement absolument convergente. On aura ainsi

$$\lim_{U\to\infty} \int_{-\infty}^{+\infty} e^{-x^2+y^2} H_m(y+v) H_n(y-v) \frac{\sin U(y-x)}{\pi(y-x)}\, dy,$$

ce qui admet, comme il est bien connu, la valeur $H_m(x+v) H_n(x-v)$, et par cela la démonstration de l'équation intégrale (62) est achevée.

Pour $v=0$, les relations (61) et (62) se réduisent à (19) et (21).

Observons que ces deux relations (61) et (62) permettent d'écrire l'équation intégrale suivante:

$$\left.\begin{aligned} &\frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(u+ix)^2-(v+iy)^2} (v+iy)^{m-n} L_n^{(m-n)} (2u^2+2v^2)\, du\, dv \\ &\qquad = (-1)^n (x-iy)^{m-n} L_n^{(m-n)} (2x^2+2y^2), \qquad (m \geqq n) \end{aligned}\right\} . \quad (63)$$

et cette relation pourra être simplifiée de façon à redonner une forme particulière de l'équation de WILSON (12c).

Une équation du même genre est

$$2^m n! \frac{1}{\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(u+ix)^2-(v+ix)^2} (iu)^{m-n} L_n^{(m-n)} (2uv)\, du\, dv = (-1)^n H_{m+n}(x) \,. \quad (64)$$

qui peut être démontrée directement à l'aide de (24′).

Si nous remplaçons, dans (61), les variables auxiliaires u et v par

$\varrho \cos\theta$ et $\varrho \sin\theta$, nous serons conduits, eu égard à (19), à la relation:

$$\left.\begin{aligned} &\int_{-\infty}^{+\infty} e^{-(x-i\varrho)^2} H_m(x)\, H_n(x)\, dx = \\ &\qquad = e^{(m-n)i\theta} \int_{-\infty}^{+\infty} e^{-(x-i\varrho\cos\theta)^2} H_m(x+\varrho\sin\theta)\, H_n(x-\varrho\sin\theta)\, dx \end{aligned}\right\} . \quad (65)$$

où ϱ et θ sont arbitraires. Pour $\varrho = 0$, on retrouve la propriété d'orthogonalité des polynomes d'HERMITE; pour $\theta = 0$, (65) devient une identité. Remarquons que le second membre de (65) est indépendant du paramètre θ. En particulier, pour $\theta = \frac{\pi}{2}$, nous retrouvons la relation (20′) du § 3. Nous allons voir aussi l'analogue de (65) pour les développements en série.

Remarquons que les formules inverses (19) et (21), écrites avec les fonctions de WHITTAKER correspondantes [28]), prennent une forme très symétrique qui est peut-être susceptible de généralisations. Par exemple, si $m+n$ est un entier pair, on aura les formules inverses

$$\int_0^\infty J_{-\frac{1}{2}}(2\sqrt{uv})\, u^{-\frac{3}{4}} W_{\frac{m}{2}+\frac{1}{4}, \pm\frac{1}{4}}(u)\, W_{\frac{n}{2}+\frac{1}{4}, \pm\frac{1}{4}}(u)\, du = (-1)^{\frac{m+n}{2}}\, 2^{-\frac{m+n+1}{2}}\, v^{-\frac{3}{4}} W_{\frac{m+n+1}{2}, \frac{m-n}{2}}(2v),$$

$$(-1)^{\frac{m+n}{2}}\, 2^{-\frac{m+n+1}{2}} \int_0^\infty J_{-\frac{1}{2}}(2\sqrt{uv})\, u^{-\frac{3}{4}} W_{\frac{m+n+1}{2}, \frac{m-n}{2}}(2u)\, du = v^{-\frac{3}{4}} W_{\frac{m}{2}+\frac{1}{4}, \pm\frac{1}{4}}(v)\, W_{\frac{n}{2}+\frac{1}{4}, \pm\frac{1}{4}}(v).$$

Si $m+n$ est impair, on aura des équations intégrales analogues, avec la fonction de BESSEL d'ordre $+\frac{1}{2}$.

2°. *Applications.* Nous avons vu [29]) que les polynomes de LAGUERRE vérifient la relation

$$\sum_{k=0}^{n} L_k^{(\alpha)}(x)\, L_{n-k}^{(\beta)}(y) = L_n^{(\alpha+\beta+1)}(x+y) = \sum_{k=0}^{n} L_k^{(\alpha+\beta)}(x+y). \quad . \quad . \quad (66)$$

Cette formule, par spécialisation des paramètres α et β, nous donnera des résultats intéressants. Posons, par exemple, $\alpha = \beta = -\frac{1}{2}$. Si nous rappelons les relations (15) qui lient les polynomes d'HERMITE et de LAGUERRE, nous aurons la généralisation d'une formule antérieure:

$$\sum_{k=0}^{n} \binom{n}{k} H_{2k}(x)\, H_{2n-2k}(y) = (-1)^n\, 2^{2n}\, n!\, L_n(x^2+y^2) \quad . \quad . \quad (67)$$

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28) Modern Analysis (cité sous [11])).

29) FELDHEIM, [8].

En appliquant aux deux membres la transformation $G_x(2, -iz)$, les relations (3c) et (62) donneront:

$$2^{-n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} H_{2k}(z\sqrt{2})\, H_{2n-2k}(y\sqrt{2}) = H_n(y+z)\, H_n(y-z) \quad . \quad (68)$$

et cette formule est une conséquence immédiate de la formule de MEHLER[30]. La relation (67), si l'on y applique la transformation $G_x(2, -iz)$, redonne en vertu de (3c) et (68), l'équation intégrale (62). Nous avons supposé évidemment, pour que cette remarque ait un sens, que (68) a été démontré directement, sans l'emploi de (62). Elle permet encore d'écrire une relation analogue à (65):

$$\left.\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} H_{2k}(\varrho)\, H_{2n-2k}(\varrho) = \\ = \sum_{k=0}^{n} \binom{n}{k} H_{2k}(\varrho\sqrt{2}\cos\theta)\, H_{2n-2k}(\varrho\sqrt{2}\sin\theta) \end{aligned}\right\} \quad . \quad . \quad (69)$$

et l'on voit que le second membre est indépendant de θ.

Si l'on fait ensuite, dans (66), $\alpha = -\frac{1}{2}$, $\beta = 0$, les mêmes formules (15) donneront lieu à la relation

$$\frac{H_{2n+1}(x)}{2x} = \sum_{k=0}^{n} (-1)^k\, 2^{2k}\, k! \binom{n}{k} L_k(x^2-y^2)\, H_{2n-2k}(y) \quad . \quad . \quad (70)$$

(y étant arbitraire). L'application de (62) à cette égalité conduit à

$$\left.\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-(x+iz)^2} \frac{H_{2n+1}(x\sqrt{2})}{2x}\, dx = \\ = \sum_{k=0}^{n} 2^k \binom{n}{k} H_k(y+z)\, H_k(y-z)\, H_{2n-2k}(iy\sqrt{2}) \end{aligned}\right\} \quad . \quad . \quad (71)$$

Le second membre représente ici encore une expression qui est indépendante de y. Si l'on fait $y = 0$, on retrouve une relation établie déjà dans le n°. 3 du § 2 [31].

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[30]) Voir § 2, n°. 2. La relation (68) a été récemment démontrée par F. SCHOBLIK, Ueber eine Funktionalbeziehung Hermitescher Polynome. Monatsh. Math. Phys. **47**, 333—337 (1939).

[31]) Un grand nombre des relations liant les polynomes d'Hermite et de Laguerre, parmi lesquelles, par exemple, les formules déduites de (31), et la relation (38), ont été récemment données par L. TOSCANO, Numeri di Stirling generalizzati, etc. Comm. Pontif. Acad. Sc. **3**, 721—757 (1939), Mémoire paru après que le présent travail a été rédigé. La méthode de l'article cité est entièrement différente de la nôtre, et fait usage d'opérateurs différentiels, appliqués à

$$\frac{d^n(e^{-x^2})}{dx^n} = (-1)^n e^{-x^2} H_n(x), \quad \text{et à} \quad \frac{d^n(x^{n+\alpha} e^{-x})}{dx^n} = n!\, x^\alpha e^{-x} L_n^{(\alpha)}(x).$$

Continuons encore à considérer des conséquences de la formule (66), où nous supposons maintenant les paramètres α et β des entiers non-négatifs (que nous désignerons, pour distinction, par r et s), et remplaçons x et y respectivement par $2x^2$ et $2y^2$. Appliquons à cette relation, simultanément pour x et y, la transformation (21). Il vient

$$\left.\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k} H_k(u)\, H_{k+r}(u)\, H_{n-k}(v)\, H_{n-k+s}(v) = \\ &= 2^{n+r+s}\, n!\, \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x+iu)^2-(y+iv)^2}(ix)^r (iy)^s L_n^{(r+s+1)}(2x^2+2y^2)\, dx\, dy\end{aligned}\right\} \quad (72)$$

r et s étant des entiers arbitraires, non-négatifs. Il ne sera pas superflu d'indiquer le cas particulier de (72) correspondant au cas le plus simple où $r = s = 0$. Alors (72) devient, en vertu de (63),

$$\sum_{k=0}^{n}\binom{n}{k} H_k^2(u)\, H_{n-k}^2(v) = 2^n\, n!\sum_{k=0}^{n}(-1)^k L_k(2u^2+2v^2), \ldots . \quad (73)$$

et l'on vérifie que cette relation peut aussi être déduite de la formule de MEHLER.

Nous insérons ici un résultat pouvant être déduit aussi de (67), en y posant $x = u + v$, $y = u - v$, et appliquant aux deux membres les transformations (61) et (62). Nous n'écrivons ce résultat, pour plus de simplicité, que dans le cas où $n = 2m$:

$$\left.\begin{aligned}&\binom{2m}{m}(2m)!\, L_{2m}(\varrho^2) + 2\sum_{k=1}^{m}(-1)^k\binom{2m}{m-k}(2m-2k)!\, \varrho^{4k}\cos 4k\theta \,.\, L_{2m-2k}^{(4k)}(\varrho^2) = \\ &\qquad\qquad = H_{2m}(\varrho\cos\theta)\, H_{2m}(\varrho\sin\theta).\end{aligned}\right\} \quad (74)$$

Les cas particuliers correspondant à $\theta = 0$ et $\theta = \frac{\pi}{4}$ de cette relation sont à confronter aux relations établies dans I (§ 2, n⁰. 2).

3⁰. *Développement du produit de deux polynomes d'*HERMITE *de degrés et d'arguments différents.* Passons à la généralisation de la formule (68) en cherchant le développement analogue pour produits de polynomes d'HERMITE de degrés différents. Nous partons de la fonction génératrice (2) qui permet d'écrire que

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{t^m\, T^n}{m!\, n!} H_m(x+y)\, H_n(x-y) = \\ &\qquad = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(t+T)^r\,(t-T)^s}{r!\, s!}\, 2^{-\frac{r+s}{2}} H_r(x\sqrt{2})\, H_s(y\sqrt{2}),\end{aligned}$$

et il faut chercher le coefficient de $t^m\, T^n$ dans la somme double du second membre.

Indiquons immédiatement la relation obtenue pour $T=0$:

$$H_m\left(\frac{x+y}{2}\right)=\sum_{k=0}^{m}\binom{m}{k}\mathcal{H}_k(x)\,\mathcal{H}_{m-k}(y), \quad . \quad . \quad . \quad . \qquad (75)$$

qui constitue la généralisation de la formule (8).

En général, on a

$$\left.\begin{aligned} &H_m(x+y)\,H_n(x-y)= \\ &=(-1)^n\,2^{-\frac{m+n}{2}}\sum_{k=0}^{m+n}A_k^{(m,n)}\,H_k(x\sqrt{2})\,H_{m+n-k}(y\sqrt{2}) \\ \text{avec}\qquad & \\ &A_k^{(m,n)}=\sum_{r=0}^{k}(-1)^r\binom{n}{r}\binom{m}{k-r} \end{aligned}\right\} \quad . \qquad (76)$$

Remarquons que $A_k^{(m,n)}$ est le coefficient de x^k dans le développement du produit $(1+x)^m\,(1-x)^n$. Si $m=n$, ce coefficient n'est différent de 0 que pour k pair, et $A_{2k}^{(n,n)}=(-1)^k\binom{n}{k}$, de sorte que (76) redonne la relation (68). Si l'on fait $x=0$, on aura le développement

$$\mathcal{H}_m(y)\,\mathcal{H}_n(y)=2^{-(m+n)}\sum_{k=0}^{\left[\frac{m+n}{2}\right]}(-1)^k\,\frac{(2k)!}{k!}\,A_{2k}^{(m,n)}\,H_{m+n-2k}(y) \quad . \qquad (76')$$

qui est à comparer à la formule (16).

La relation (76) peut encore être généralisée, par exemple, de la façon suivante:

$$\left.\begin{aligned} &H_m(x\cos\alpha+y\sin\alpha)\,H_n(x\sin\alpha-y\cos\alpha)= \\ &\quad=(-1)^n\sin^m\alpha\cos^n\alpha\sum_{k=0}^{m+n}\cot g^k\alpha\,.\,B_k^{(m,n)}\,H_k(x)\,H_{m+n-k}(y) \\ \text{avec}\qquad & \\ &B_k^{(m,n)}=\sum_{r=0}^{k}(-1)^k\binom{n}{r}\binom{m}{k-r}\,tg^{2r}\,\alpha \end{aligned}\right\} \qquad (77)$$

α étant quelconque. Pour $\alpha=\frac{\pi}{4}$, (77) se réduit à (76). Indiquons une conséquence de (77). Multiplions ses deux membres par e^{-x^2}, et intégrons par rapport à x de $-\infty$ à $+\infty$. (L'intégration terme à terme, comme pour toutes les séries intervenant dans ce travail, est évidemment justifiée). Le second membre, par suite de l'orthogonalité, n'est différent de 0 que pour $k=0$, et alors on retrouve la relation (52).

4⁰. *Quelques conséquences des formules* (18). Remarquons que les relations (18) peuvent être considérées comme une généralisation de la

formule (31)[32]. En effet, si nous posons, dans (18), $n=0$, la formule fondamentale (13a) permet d'écrire que

$$(-1)^m c_\nu(m, 0, \beta-m, \alpha+m) = (-1)^\nu \binom{m+\alpha}{m-\nu},$$

de sorte que

$$L_m^{(\alpha)}(y) = \sum_{\nu=0}^{m} (-1)^{m-\nu} \binom{\beta}{m-\nu} L_\nu^{(\alpha+\beta)}(y),$$

et ceci est équivalent à (31).

Nous avons vu que l'inversion de (18), sous la forme (35), est impossible, et nous avons démontré, par contre, la possibilité du développement (60). On peut alors se proposer de chercher l'inversion de cette formule (60), c'est-à-dire une relation analogue à (18):

$$L_m^{(\alpha)}(y)\, L_n^{(\beta)}(y) = \sum_{n=0}^{m+n} h_k(m, n; \alpha, \beta)\, L_k^{(\alpha+\beta)}(2y). \quad . \quad . \quad . \quad (78)$$

Nous pouvons démontrer, par la même méthode qui nous a conduit au résultat exprimé par (18)[33], que les coefficients h_k de ce développement vérifient la relation

$$h_k(m, n; \alpha, \beta) = (-1)^{m+n-k}\, h_k(m, n; \beta-m+n, \alpha+m-n) \quad . \quad (78a)$$

(L'application de la transformation de HANKEL à (78) redonnera, en vertu de (12c), l'équation intégrale (22)).

En particulier, si $m=n$, les coefficients h_k sont nuls pour les valeurs impaires de k. Si nous prenons, par exemple, $m=n$, $\alpha=\beta=\frac{1}{2}$, et $y=v^2$, la formule (78) devient

$$H_{2n+1}^2(\sqrt{v}) = 2v \sum_{k=0}^{n} h_k\, L_{2k}^{(1)}(2v) \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

avec les valeurs correspondantes $h_k(n, n; \frac{1}{2}, \frac{1}{2})$ des coefficients. Or, si nous appliquons à cette relation la transformation de HANKEL de noyau $J_1(2\sqrt{uv})\, e^{-v} v^{-\frac{1}{2}}$, l'équation (12$c$) de WILSON donnera

$$\int_0^\infty J_1(2\sqrt{uv}) e^{-v} v^{-\frac{1}{2}} H_{2n+1}^2(\sqrt{v})\, dv = 2u^{\frac{1}{2}} \sum_{k=0}^{n} h_k L_{2k}^{(1)}(2u) = e^{-u} u^{-\frac{1}{2}} H_{2n+1}(\sqrt{u}).$$

C'est une équation intégrale établie par S. C. MITRA[34]. On déduit

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32) Cette formule (31) a été déjà donnée, comme nous venons de le voir, dans le Mémoire de E. KOGBETLIANTZ, Recherches sur la sommabilité des séries d'Hermite. Ann. Sc. de l'Ec. Norm. Sup. 3, **49**, 137—221 (1932).

33) FELDHEIM, [8].

34) MITRA, Journ. London Math. Soc. **11**, 252 (1936).

ensuite de (78), par dérivation, et en posant $\alpha = \beta = -\frac{1}{2}$, le développement

$$H_{2n-1}(\sqrt{v})\, H_{2n}(\sqrt{v}) = 2\, v^{\frac{1}{2}} \sum_{k=0}^{n-1} h'_k\, L_{2k+1}(2v), \quad . \quad . \quad . \quad (79a)$$

qui fournit, par application de la transformation de HANKEL de noyau $J_0(2\sqrt{u\,v})\, e^{-v}\, v^{-\frac{1}{2}}$, l'équation intégrale analogue à celle de MITRA, et trouvée par A. ERDÉLYI [35]). (Ces équations intégrales sont des cas particuliers de (22), et leur déduction précédente rentre dans la remarque faite au sujet de la formule (78)).

Remarquons, pour terminer, que (18) et (78) donnent lieu aux développements

$$H_{2n}(\sqrt{v}) = \sum_{k=0}^{n} h_k\, L_k^{(n-1)}(v) = (-1)^n \sum_{k=0}^{n} h_k \frac{v^k}{k!} \quad . \quad . \quad . \quad . \quad (80)$$

(avec les mêmes h_k dans les coefficients!), et

$$H_{2n}(\sqrt{v}) = \sum_{k=0}^{\left[\frac{n}{2}\right]} h'_k\, L_{2k}^{(n-1)}(2v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80')$$

L'application de la transformation de HANKEL à (80) (ainsi qu'à la relation que l'on en déduit par dérivation) ou à (80') conduira à des équations intégrales démontrées également par MITRA, et MEIJER [36]), et formant des cas particuliers de (22).

Budapest, le 12 février, 1940.

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[35]) ERDÉLYI, [2].

[36]) MITRA, Math. Zeitschr. **43**, 205—211 (1938); C. S. MEIJER, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **41**, 744—755 (1938).

Mathematics. — *On the thermo-hydrodynamics of perfectly perfect fluids.* I. By D. VAN DANTZIG. (Communicated by Prof. J. A. SCHOUTEN).

(Communicated at the meeting of February 24, 1940.)

Summary.

The equations of motion of a perfectly perfect (in particular of a relativistically perfect) fluid are brought into a general invariant form, independent of metrical geometry (§ 1). They are shown to be derivable from a simple variational principle. It states that the integral of the pressure over an arbitrary fourdimensional domain in space time, under a deformation "dragging along" (cf. § 1) the chemical parameters λ^r and the temperature vector ϑ^h, hence also the congruence of macroscopic wordlines, is equal to δx^4 times the virtual heat of the deformation, flowing through the boundary into U (§ 2). In § 3 some other variational relations are derived. In § 4 a result due to EISENHART and used by SYNGE is obtained by metrical specialisation from a one dimensional variational principle.

§ 1. *The equations of motion.*

The equations of motion of continuously distributed matter are according to EINSTEIN [1])

$$\nabla_j \mathfrak{T}^j_{.i} = 0, \; ^{2)} \qquad (1)$$

where ∇_j is the symbol for covariant derivation, whereas for a relativistically perfect fluid [3])

$$\mathfrak{T}^h_{.i} = \mathfrak{K}^h_{.i} + \mathfrak{S}^h_{.i}, \qquad (2)$$

$$\mathfrak{K}^h_{.i} = -(\varrho + \mathfrak{p})\, i^h\, i_i + \mathfrak{p}\, A^h_i, \; ^{4)}\, ^{5)}\, ^{6)} \qquad (3)$$

$$\mathfrak{S}^h_{.i} = \sqrt{-\mathfrak{g}}\,(F^{hk} F_{ik} - \tfrac{1}{4} A^h_i F_{jk} F^{jk}) \qquad (4)$$

1) E. g. A. EINSTEIN, [1], in particular § 17, 19, 20. The numbers between square brackets refer to the bibliography at the end of the paper.

2) The suffixes h, i, j, k, l run independently through the range 1, 2, 3, 4, corresponding with the space-time coordinates x^h; r, s, t run through the range $5, \ldots, 4+n$, corresponding with the n chemical components Γ^r; $\varkappa, \lambda, \mu, \nu$ through the range $1, 2, 3, \ldots, 4+n$, and A, B, C through the range $0, 1, 2, \ldots, 4+n$.

3) The remark that (3) is valid for a perfect fluid in *adiabatic* motion only, was made by EINSTEIN [1] already; it seems to have been neglected by several later autors.

4) In order to avoid superfluous factors $\sqrt{-\mathfrak{g}}$ ($\mathfrak{g} = det\, g_{ij}$), ϱ itself instead of $\varrho\sqrt{-\mathfrak{g}}$ stands for the proper energy density. All densities except ϱ, ϱ_0, ϱ are denoted by Gothic letters.

5) A^h_i is the unit-tensor of space-time: $A^h_i = \begin{cases} 1, & h = i \\ 0, & h \neq i \end{cases}$. The unit-tensors Δ^r_s, $E^\varkappa_\lambda$, $\mathfrak{A}^A_B$ play the same rôle as A^h_i with regard to their respective ranges.

6) $i^h = \dfrac{dx^h}{ds}$ is a unit-vector along the macroscopic worldlines, $i^h\, i_h = +1$, $i^4 > 0$.

Usually $\mathfrak{K}^{h}_{.i}$ is called the material part and $\mathfrak{S}^{h}_{.i}$ the electromagnetic part of the total tensordensity $\mathfrak{T}^{h}_{.i}$ of stress, momentum, and energy. According to MAXWELL's equations

$$\nabla_j \mathfrak{S}^{j}_{.i} = -F_{ik}\,\mathfrak{s}^k \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (5)$$

where $\mathfrak{s}^h$ is the electric current-density. Hence (1) becomes

$$\nabla_j \mathfrak{K}^{j}_{.i} = F_{ik}\,\mathfrak{s}^k. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (6)$$

In order to bring our notations in accordance with previous papers[7]) we add to $\mathfrak{K}^{h}_{.i}$ the potential momentum and energy of the matter with respect to the electromagnetic field, which is $\mathfrak{s}^h \varphi_i$. In fact, let the substance be a mixture of n different components, in relative equilibrium with each other, so that they all have the same velocity. Further let m^r and e^r be the mass and charge of one particle (molecule, ion, electron, etc.) of the r^{th} component, $N_r^{\mathrm{d}V}$ the number of particles[8]) of the r^{th} component in a volume-element $\mathrm{d}V$, $\mathfrak{N}^h_r$ the corresponding particle-current, so that $N_r^{\mathrm{d}V} =_{\mathrm{df}} \mathfrak{N}^h_r\, \mathrm{d}\mathfrak{B}_h$ [9]), $\mathrm{d}\mathfrak{B}_i$ being the components of $\mathrm{d}V$[10]), and f^r_i the potential momentum and energy of each particle of the r^{th} component. Then $\mathfrak{s}^h = \mathfrak{N}^h_r e^r$, $f^r_i = e^r \varphi_i$. Hence the total potential momentum and energy in $\mathrm{d}V$ with respect to the field is

$$N_r^{\mathrm{d}V} f^r_i = \mathfrak{N}^h_r\, \mathrm{d}\mathfrak{B}_h \,.\, e^r \varphi_i = \mathfrak{s}^h\, \mathrm{d}\mathfrak{B}_h \,.\, \varphi_i.$$

The total stress-tensor of the matter is therefore

$$\mathfrak{P}^{h}_{.i} =_{\mathrm{df}} \mathfrak{K}^{h}_{.i} + \mathfrak{N}^h_r f^r_i = \mathfrak{K}^{h}_{.i} + \mathfrak{s}^h \varphi_i, \quad . \; . \; . \; . \; . \; . \qquad (7)$$

whereas the remaining part is

$$\mathfrak{T}^{h}_{.i} - \mathfrak{P}^{h}_{.i} = \mathfrak{S}^{h}_{.i} - \mathfrak{s}^h \varphi_i. \quad . \; . \; . \; . \; . \; . \; . \; . \qquad (8)$$

If the mixture is neutral, the second term in the right members of (7) and (8) vanishes.

Introducing (7) into (6) we obtain

$$\nabla_j \mathfrak{P}^{j}_{.i} = \mathfrak{N}^j_r \nabla_i f^r_j, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (9)$$

where we used MAXWELL's equations in the form

$$F_{ik} = 2\nabla_{[i}\varphi_{k]} =_{\mathrm{df}} \nabla_i \varphi_k - \nabla_k \varphi_i,\ \nabla_j F^{hj}\sqrt{-\mathfrak{g}} = \mathfrak{s}^h, \text{ whence } \nabla_j \mathfrak{s}^j = 0.$$

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7) D. VAN DANTZIG, [2], [3], [4], referred to as Ph. Th., R. Th., and R. G. respectively.

8) If we wish to avoid any molecular model, we may read $N_r^{\mathrm{d}V}$ as the number of *gm. mol.* of the r[th] component, contained in $\mathrm{d}V$, and $\mathfrak{N}^h_r$ as the corresponding current-density. Then λ^r, η^r, π^r_i, f^r_i, m^r, e^r also have to be reckoned pro gm. mol. instead of pro molecule. N.B. Summation convention, for r, s also.

9) By $=_{\mathrm{df}}$ and ${}_{\mathrm{df}}=$ we denote an equality defining its *left* and its *right* member respectively.

10) Cf. Ph. Th. p. 684.

Equation (9) remains valid if other than electromagnetic forces work upon the matter, provided they are derivable from a vector potential, which is the same for all particles of each component of the mixture, and if we take for f_i^r the sum of $e^r \varphi_i$ and this vector potential.

According to (3), (7) the mixture is what I called a "perfectly perfect fluid" [11]), the stress-tensor and particle-densities having the form [12])

$$\mathfrak{P}^{h}_{.i} = \vartheta^h \mathfrak{p}_i + \mathfrak{p} A_i^h, \quad \ldots \ldots \ldots \quad (10)$$

$$\mathfrak{N}^h_r = \mathfrak{p}_r \vartheta^h, \quad \ldots \ldots \ldots \ldots \quad (11)$$

where $\vartheta^h = i^h \vartheta_0 = c\, i^h / k T_0$ [13]) is the "temperature-vector", T_0 being the proper temperature and k BOLTZMANNs constant, whereas [6])

$$\mathfrak{p}_i = - c^{-1} k T_0 (\varrho + \mathfrak{p}) i_i + \mathfrak{p}_r f_i^r. \quad \ldots \ldots \quad (12)$$

According to Ph. Th. p. 697, R. Th. p. 604, generalised for a mixture of n components. $\mathfrak{p}$ can for constant x^h (i.e. in each given point of space-time) be considered as a function of $n+4$ independent variables ϑ^h and λ^r with

$$\mathfrak{p}_s = \frac{\partial \mathfrak{p}}{\partial \lambda^s}, \qquad \mathfrak{p}_i = \frac{\partial \mathfrak{p}}{\partial \vartheta^i}. \quad \ldots \ldots \ldots \quad (13)$$

If the dependance upon the coordinates is taken into account also, $\mathfrak{p}$ depends upon them: 1^0. through the ϑ^h and λ^r, 2^0. through the f_i^r, which enter into $\mathfrak{p}$ in the combination $\lambda^r + f_j^r \vartheta^j$ only, 3^0. through the g_{ij}. If the curvature is negligible and no direction in empty space is given, $\mathfrak{p}$ will have the form [14])

$$\mathfrak{p} = \varphi (\vartheta_0, \lambda^r + f_i^r \vartheta^i) \sqrt{-\mathfrak{g}}, \qquad \vartheta_0 =_{\mathrm{df}} \vartheta^h i_h = \sqrt{g_{ij} \vartheta^i \vartheta^j} = c/kT_0. \quad (14)$$

We suppose φ to be a function of its $n+1$ arguments alone (which

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11) Ph. Th. p. 686; R. Th. p. 602.

12) A very thorough discussion of the relativistic stress-tensor on an axiomatic base was given by J. L. SYNGE, [1]. I find however some difficulty in accepting SYNGE's hypothesis of "elementary impulses". For either the "gas" consisting of these elementary impulses would behave in a way entirely different from any other gas with respect to its thermodynamic properties, or otherwise it could, like a photon-gas, be considered like one of the components of our mixture, admitted the alterations mentioned in R. G. § 10. In that case however it would be difficult to see, how the elementary impulses could have any appreciable influence, except at very high temperatures. In any case the complete independence of the interaction between molecules and the temperature would be hard to account for.

13) In order to avoid factors c in the general (non-metrical) part of the theory, where they have nothing to do, we may take with respect to orthogonal coordinates $x^4 = t$ (instead of $x^4 = ct$), $g_{44} = c^2$, $g^{44} = c^{-2}$, etc. This leads to some unusual factors c, but makes comparison with the non-metrical as well as the classical theory easier. This system of notations was introduced by DE DONDER [1].

14) Cf. e.g. for an ideal gas R. G. (51).

themselves are functions of x^h) and not to depend upon the x^h explicitly. Then, putting $\varphi' =_{\mathrm{df}} \partial\varphi/\partial\vartheta_0$, $\varphi_r =_{\mathrm{df}} \partial\varphi/\partial\lambda^r$,

$$\left.\begin{aligned} \mathfrak{p}_r &= \varphi_r \sqrt{-\mathfrak{g}}, \\ \mathfrak{p}_i &= (\varphi_r f_i^r + \varphi' i_i) \sqrt{-\mathfrak{g}}, \\ \frac{\partial \mathfrak{p}}{\partial f_j^r} &= \vartheta^j \varphi_r \sqrt{-\mathfrak{g}} = \mathfrak{N}_r^j, \\ \frac{\partial \mathfrak{p}}{\partial g_{ij}} &= \mathfrak{p}\, g^{ij} + \tfrac{1}{2} \varphi' \sqrt{-\mathfrak{g}} \,.\, \vartheta^i \vartheta^j / \vartheta_0. \end{aligned}\right\} \quad \ldots \ldots \quad (15)$$

Denoting by ∂_i^0 the operator $\partial/\partial x^i$ under constant ϑ^h and λ^r, and by ∂_i the complete operator $\partial/\partial x^i$, so that

$$\partial_i^0 =_{\mathrm{df}} \partial_i - (\partial_i \vartheta^h) \frac{\partial}{\partial \vartheta^h} - (\partial_i \lambda^r) \frac{\partial}{\partial \lambda^r}, \quad \ldots \ldots \quad (16)$$

a short calculation shows that (9) is equivalent with

$$\boxed{\partial_j \mathfrak{P}^j_{.i} - \partial_i^0 \mathfrak{p} = 0} \quad \ldots \ldots \ldots \quad (17)$$

In this equation the metrical quantities do not occur anymore. Though its terms themselves are not invariant under arbitrary transformations of coordinates in space-time, the equation (17) is invariant as a whole, $\mathfrak{P}^h_{.i}$ having the form (10). At the other hand it can be shown that it is also necessary that $\mathfrak{P}^h_{.i}$ has the form (10), i.e. that the fluid is perfectly perfect, in order that (17) be invariant, as long as the operator ∂_i^0 is defined by (16). We may therefore consider (17) as the equations of motion of an arbitrary perfectly perfect fluid, independent of metrical geometry.

From here on, except when the contrary is stated, all equations are independent of any special assumption concerning the relation between energy and momentum. Hence they are independent of the axioms of relativity theory and also of metrical geometry.

Inserting (10) and (16) into (17) we obtain

$$\vartheta^j \partial_j \mathfrak{p}_i + \mathfrak{p}_j \partial_i \vartheta^j + \mathfrak{p}_i \partial_j \vartheta^j = -\mathfrak{p}_r \partial_i \lambda^r. \quad \ldots \ldots \quad (18)$$

Both sides of this equation are invariant quantities.

The differential equations

$$\frac{dx^h}{d\theta} = \vartheta^h(x) \quad \ldots \ldots \ldots \quad (19)$$

define a one-parametric group over a part of space-time. Its infinitesimal

transformation is determined by $dx^h =_{\mathrm{df}} d\theta\, \vartheta^h$, $d\theta$ being an infinitesimal increasement of the parameter θ. Its LIE-symbol is $\frac{d}{d\theta} =_{\mathrm{df}} \vartheta^i \partial_i$.

The infinitesimal transformation transforms any *scalar function* f into $f + df/d\theta$. The corresponding invariant operator working upon a general geometric object $X^{\cdot}_{\cdot}$ (where the big points stand for rows of indices) was first defined by ŚLEBODZIŃSKI [15]. It will be called the LIE-derivative and denoted by $\frac{d_L}{d\theta}$. It is defined as follows. The components of $X^{\cdot}_{\cdot} + d_L X^{\cdot}_{\cdot}$ in a point (with respect to the system of coordinates under consideration) are the components of the value, $X^{\cdot}_{\cdot}$ takes in a point $x^h + dx^h = x^h + \vartheta^h d\theta$ with respect to the system of coordinates, obtained by "dragging along" the original coordinates along dx^h, i.e. by ascribing to each point y^h as new coordinates $y^{h'}$ the original coordinates of the point from which it was obtained by the infinitesimal transformation: $y^{h'} = y^h - \vartheta^h(y)\, d\theta$. By this definition $\frac{d_L}{d\theta}$ becomes an invariant operator. Applied to a scalar f we get $\frac{d_L f}{d\theta} = \frac{df}{d\theta}$. Applied to a scalar density (or W-density [16])) $\mathfrak{p}$ of weight $+1$, to a contravariant vector v^h and to a covariant vector w_i we obtain [17])

$$\frac{d_L \mathfrak{p}}{d\theta} = \partial_j\, \mathfrak{p}\, \vartheta^j = \frac{d\mathfrak{p}}{d\theta} + \mathfrak{p}\, \partial_j \vartheta^j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)^{18)}$$

$$\frac{d_L v^h}{d\theta} = \vartheta^j \partial_j v^h - v^j \partial_j \vartheta^h = \frac{dv^h}{d\theta} - v^j \partial_j \vartheta^h, \quad . \quad . \quad . \quad (21)^{19)}$$

$$\frac{d_L w_i}{d\theta} = 2\vartheta^j \partial_{[j} w_{i]} + \partial_i \vartheta^j w_j = \frac{dw_i}{d\theta} + w_j \partial_i \vartheta^j \quad . \quad . \quad . \quad (22)$$

Evidently

$$\frac{d_L \vartheta^h}{d\theta} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

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15) W. ŚLEBODZIŃSKI [1]. Cf. also J. A. SCHOUTEN and E. R. VAN KAMPEN [1], D. VAN DANTZIG [5]; J. A. SCHOUTEN and D. J. STRUIK [1], p. 142.

16) W-densities are quantities which under transformations of coordinates are multiplied by a power of the *absolute value* of the transformation-modulus. They were introduced by H. WEYL, e. g. [1] p. 98. Cf. also J. A. SCHOUTEN [1], J. A. SCHOUTEN and D. VAN DANTZIG [1].

17) Cf. J. A. SCHOUTEN and E. R. VAN KAMPEN, loc. cit., p. 4.

18) A differential operator (e. g. ∇_j, ∂_j, $\frac{d_L}{d\theta}$, etc.) works upon all following quantities untill either a closing bracket belonging to an opening bracket preceding the operator, or a $+$, $-$ or $=$ sign, or the end of the formula is met. The symbol for a differential, a variation or an element (e. g. d, δ, d) belongs to the immediately following symbol alone.

19) Introducing the Lie-symbols $X_1 =_{\mathrm{df}} \frac{d_L}{d\theta} =_{\mathrm{df}} \vartheta^j \partial_j$ and $X_2 =_{\mathrm{df}} v^j \partial_j$, we have $\left(\frac{d_L}{d\theta} v^j\right) \partial_j = (X_1, X_2)$, where the right member is the ordinary LIE-bracket.

Applied to a covariant vectordensity (or vector-W-density) $\mathfrak{p}_i$ of weight $+1$ we obtain

$$\frac{d_{\mathrm{L}}\,\mathfrak{p}_i}{d\theta} = \vartheta^j\,\partial_j\,\mathfrak{p}_i + \mathfrak{p}_j\,\partial_i\,\vartheta^j + \mathfrak{p}_i\,\partial_j\,\vartheta^j, \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

which is the left side of (18). Hence we have proved:

Theorem 1. *The equations of motion* (17) *are equivalent with the system of equations*

$$\boxed{\frac{d_{\mathrm{L}}\,\mathfrak{p}_i}{d\theta} = -\,\mathfrak{p}_r\,\partial_i\,\lambda^r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

§ 2. *Derivation from a variational principle.*

In this paragraph we show that the equations of motion (17) for a perfectly perfect fluid can be derived from a simple variational principle.

Let U be a part of space-time, bounded by a differentiable threedimensional manifold B. The latter is provided with an exterior orientation [20], viz the direction *outward* from U. An element of U will be denoted by $\mathrm{d}\,U$, its fourdimensional volume (as measured by the coordinates under consideration) by $\mathrm{d}\,\mathfrak{U}$; an element of B by $\mathrm{d}B$ or $\mathrm{d}\,V$; its components by $\mathrm{d}\mathfrak{V}_i$ or $\mathrm{d}\mathfrak{B}_i$. Let further a contravariant vectorfield $z^h = z^h(x^i)$ be defined in U, the components of which have continuous first derivatives with respect to the x^i.

In order to determine easily the variations of several quantities it is useful to introduce the LIE-variation δ_{L} with respect to the equations $dx^h/d\mu = z^h(x)$ in the same way as $d_{\mathrm{L}} = d\theta\,\dfrac{d_{\mathrm{L}}}{d\theta}$ was defined with respect to (19).

The variation of an integral $W =_{\mathrm{df}} \int\limits_U \mathfrak{W}\,\mathrm{d}\,\mathfrak{U}$ under the infinitesimal transformation $\delta =_{\mathrm{df}} \delta\,x^j\,\partial_j =_{\mathrm{df}} \delta\,\mu\,.\,z^j\,\partial_j$ is then

$$\delta\,W = \delta_{\mathrm{L}}\,W = \int\limits_U (\delta_{\mathrm{L}}\,\mathfrak{W})\,\mathrm{d}\,\mathfrak{U} = \int\limits_U (\partial_j\,\mathfrak{W}\,\delta x^j)\,\mathrm{d}\,\mathfrak{U} = \int\limits_B \mathfrak{W}\,\delta x^j\,\mathrm{d}\,\mathfrak{V}_j, \qquad (26)$$

as by definition $\delta_{\mathrm{L}}\,\mathrm{d}\,\mathfrak{U} = 0$, the new element $\mathrm{d}\,U$ being obtained by "dragging along" the original element. It is of course trivial that $\delta\,W$ reduces to a boundary integral as the variation consists of a displacement alone.

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20) An exterior orientation of a simplex Σ is an orientation for each simplex of complementary dimension having exactly one point in common with Σ; Σ itself need *not* be oriented. Cf. O. VEBLEN and J. H. C. WHITEHEAD [1], p. 56, and J. A. SCHOUTEN and D. VAN DANTZIG [1].

Now we take in particular $\mathfrak{W} =_{\mathrm{df}} \mathfrak{p} = \mathfrak{p}(x^h, \vartheta^h, \lambda^r)$. In U a congruence of curves is defined, viz the macroscopic wordlines of the motion, satisfying the equations (17). We now define the operator $\overset{0}{\delta}_{\mathrm{L}}$ working upon a quantity $X^{\cdot}_{\cdot} =_{\mathrm{df}} X^{\cdot}_{\cdot}(x^h, \vartheta^h, \lambda^r)$, depending upon ϑ^h and λ^r, by requiring that not only the coordinates, but also the values of ϑ^h and λ^r shall be "dragged along". More precisely: $X^{\cdot}_{\cdot} + \overset{0}{\delta}_{\mathrm{L}} X^{\cdot}_{\cdot}$ shall be the components with respect to the coordinates mentioned above of $X^{\cdot}_{\cdot}('x^h, '\vartheta^h, '\lambda^r)$, where $'x^h = x^h + \delta x^h$, whereas $'\vartheta^h$ and $'\lambda^r$ are defined by requiring $\overset{0}{\delta}_{\mathrm{L}} \vartheta^h = 0$, $\overset{0}{\delta}_{\mathrm{L}} \lambda^r = 0$. Hence

$$\overset{0}{\delta}_{\mathrm{L}} X^{\cdot}_{\cdot} =_{\mathrm{df}} \delta_{\mathrm{L}} X^{\cdot}_{\cdot} - \frac{\partial X^{\cdot}_{\cdot}}{\partial \vartheta^h} \delta_{\mathrm{L}} \vartheta^h - \frac{\partial X^{\cdot}_{\cdot}}{\partial \lambda^r} \delta_{\mathrm{L}} \lambda^r \quad . \quad . \quad . \quad . \quad (27)$$

In particular

$$\left.\begin{aligned} \overset{0}{\delta}_{\mathrm{L}} \mathfrak{p} &= \delta_{\mathrm{L}} \mathfrak{p} - \mathfrak{p}_i \, \delta_{\mathrm{L}} \vartheta^i - \mathfrak{p}_r \, \delta_{\mathrm{L}} \lambda^r = \\ &= \delta \mathfrak{p} + \mathfrak{p} \, \partial_j \, \delta x^j - \mathfrak{p}_i \, \delta \vartheta^i + \mathfrak{p}_i \, \vartheta^j \, \partial_j \, \delta x^i - \mathfrak{p}_r \, \delta \lambda^r = \\ &= \delta^0 \mathfrak{p} + \mathfrak{P}^{j}_{\cdot i} \, \partial_j \, \delta x^i, \end{aligned}\right\} \quad . \quad (28)$$

where $\delta^0 =_{\mathrm{df}} \delta x^i \, \overset{0}{\partial}_j$, $\overset{0}{\partial}_j$ being defined by (16).

Hence

$$\left.\begin{aligned} \overset{0}{\delta}_{\mathrm{L}} \int_U \mathfrak{p} \, d\mathfrak{U} &= \int_U (\delta x^i \, \overset{0}{\partial}_i \mathfrak{p} + \mathfrak{P}^{j}_{\cdot i} \, \partial_j \, \delta x^i) \, d\mathfrak{U} = \\ &= \int_B \mathfrak{P}^{h}_{\cdot i} \, \delta x^i \, d\mathfrak{B}_h + \int_U \delta x^i \, (\overset{0}{\partial}_i \mathfrak{p} - \partial_j \, \mathfrak{P}^{j}_{\cdot i}) \, d\mathfrak{U}. \end{aligned}\right\} \quad . \quad . \quad . \quad (29)$$

Hence we have proved:

Theorem 2. The system of equations of motion (17) *is equivalent with the variational principle*

$$\overset{0}{\delta}_{\mathrm{L}} \int_U \mathfrak{p} \, d\mathfrak{U} = \int_B P_i^{dB} \, \delta x^i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$ [21]

for any part U of space-time and any continuously differentiable variation δx^i, and also with the variational principle

$$\overset{0}{\delta}_{\mathrm{L}} \int_U \mathfrak{p} \, d\mathfrak{U} = 0$$

for any part U of space-time and any continuously differentiable [22] *variation δx^i, vanishing at the boundary B of U.*

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[21] $P_i^{dB} =_{\mathrm{df}} d\,\mathfrak{B}_h \, \mathfrak{P}^{h}_{\cdot i}$ is the amount of energy and momentum contained in or flowing through the element dB. Analogously N_r^{dV}, S^{dV}, etc. Cf. Ph. Th. p. 684.

[22] These conditions will not be mentioned explicitly further on.

This theorem seems to be sufficiently remarkable, to restate it in more explicite and intuitive terms. Let $I_U[\mathfrak{p}]$ be the integral of the pressure $\mathfrak{p}$ over a fourdimensional domain U in space time. A small deformation $x^h \rightarrow {}'x^h =_{\mathrm{df}} x^h + \delta x^h$ deforms U into a new domain $'U$. Moreover the congruence of macroscopic worldlines in U is deformed into another congruence in $'U$. If the parameter θ, defined by $\vartheta^h(x) = dx^h/d\theta$, is kept constant, the vectorfield $\vartheta^h(x)$ passes into a new vectorfield $'\vartheta^h('x) =_{\mathrm{df}} \vartheta^h(x) + d(\delta x^h)/d\theta$, which differs from $\vartheta^h('x) = \vartheta^h(x) + \delta\vartheta^h$ by $d_{\mathrm{L}}\,\delta x^h$. Leaving the *values* of λ^r unaltered, i.e. taking $'\lambda^r('x) =_{\mathrm{df}} \lambda^r(x) = \lambda^r('x) - \delta\lambda_r$, the variation of the pressure integral is the difference of the integral of $'\mathfrak{p} =_{\mathrm{df}} \mathfrak{p}('\vartheta^h('x), ('\lambda^r('x), 'x^h)$ over $'U$ and the original integral over U. This variation $\delta^0 I = I_{'U}['\mathfrak{p}] - I_U[\mathfrak{p}]$ is according to Theorem 2 (if the coordinates are such that x^4 may be interpreted as the time) equal to δx^4 times the virtual heat belonging to the deformation, flowing through te boundary *into* U, if and only if the equations of motion are satisfied.

§ 3. *Other variational relations.*

The variational principle (30) can be brought into another form. Therefore let S_0 be a part of a threedimensional manifold in space-time, which has at most one point in common with any (macroscopic) worldline, and which nowhere is tangent to such a worldline [22], and let O_0 be its twodimensional boundary. The threedimensional manifold S_θ will be defined as the locus of all points x which have a difference of "thermasy" with S_0 equal to θ, i.e. such that the worldline through x intersects S_0 in a point x_0 and that the integral

$$\int_{x_0}^{x} d\theta = \int k\,T\,dt = \int c^{-1}\,k\,T_0\,ds \quad . \quad . \quad . \quad . \quad . \quad (32)$$

taken along the worldline through x is equal to θ. Let U be the locus of all S_θ for $0 \leqq \theta \leqq \theta_1$. Its boundary consists of S_0, S_{θ_1} and the locus B' of the parts of all worldlines through O_0 between S_0 and S_{θ_1}.

We suppose U to be divided into elements $\mathrm{d}U$ of the same nature as U itself, i.e. bounded by parts of two hypersurfaces S_θ and $S_{\theta+d\theta}$ and by ∞^2 parts of worldlines passing through the boundary of an element $\mathrm{d}V$ of S_θ. Then

$$\mathrm{d}\mathfrak{U} = dx^h\,\mathrm{d}\mathfrak{V}_h = \vartheta^h\,\mathrm{d}\mathfrak{V}_h\,d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

and

$$\int_U \mathfrak{p}\,\mathrm{d}\mathfrak{U} = \int_0^{\theta_1} d\theta \int_{S_\theta} \mathrm{d}\mathfrak{V}_h\,\vartheta^h\,\mathfrak{p} = \int_0^{\theta_1} d\theta \int_{S_\theta} Z^{\mathrm{d}V}, \quad . \quad . \quad . \quad (34)$$

or

$$\boxed{\int_{S_\theta} Z^{\mathrm{d}V} = \frac{d}{d\theta} \int_{U_\theta} \mathfrak{p}\, \mathrm{d}\mathfrak{U}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

if U_θ is the part of space-time bounded by B' and S_0, S_θ.

Now let a deformation δx^h of S_0 *in itself* be given, so that

$$\delta x^i \, \mathrm{d}\mathfrak{B}_i = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

for any element $\mathrm{d}\mathfrak{B}_i$ of S_0 and let δx^h vanish at the boundary O_0 of S_0. Let every element $\mathrm{d}\mathfrak{B}_i$ of S_0 as well as the displacement δx^h be „dragged along" by the transformation d_L, so that

$$d_{\mathrm{L}} \, \delta x^h = - \delta_{\mathrm{L}} \, \vartheta^h \, d\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$ [23]

and $d_{\mathrm{L}} \, \mathrm{d}\mathfrak{B}_i = 0$. Then (36) remains valid for every element of every S_θ. Moreover $\delta x^h = 0$ on B'.

By (29) and (37) we have then $\overset{0}{\delta}_{\mathrm{L}} \mathfrak{p} = \delta_{\mathrm{L}} \mathfrak{p} - \mathfrak{p}_r \, \delta_{\mathrm{L}} \lambda^r$, hence

$$\mathrm{d}\mathfrak{U} \, \overset{0}{\delta}_{\mathrm{L}} \, \mathfrak{p} = \mathrm{d}\mathfrak{U} \, \delta_{\mathrm{L}} \, \mathfrak{p} - d\theta \, N_r^{\mathrm{d}V} \, \delta_{\mathrm{L}} \lambda^r \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

hence, using (37) again,

$$\delta^0 \int_U \mathfrak{p} \, \mathrm{d}\mathfrak{U} = \delta \int_U \mathfrak{p} \, \mathrm{d}\mathfrak{U} - \int_0^{\theta_1} d\theta \int_{S_\theta} N_r^{\mathrm{d}V} \, \delta \lambda^r. \quad . \quad . \quad . \quad (39)$$

By (26) and the boundary conditions however

$$\delta \int_U \mathfrak{p} \, \mathrm{d}\mathfrak{U} = \int_B \mathfrak{p} \, \delta x^i \, \mathrm{d}\mathfrak{B}_i = \left[\int_{S_\theta} \mathfrak{p} \, \delta x^i \, \mathrm{d}\mathfrak{B}_i \right]_0^{\theta_1} = 0. \quad . \quad . \quad (40)$$

The right member of (30) need also be extended over S_0 and S_{θ_1} only, so that we have proved:

Theorem 3. The equations of motion (17) are equivalent with the variational principle

$$- \int_0^{\theta_1} d\theta \int_{S_\theta} N_r^{\mathrm{d}V} \, \delta \lambda^r = \left[\int_{S_\theta} P_i^{\mathrm{d}V} \, \delta x^i \right]_0^{\theta_1} \quad . \quad . \quad . \quad . \quad (41)$$

for every variation δx^i of each S_θ in itself, vanishing at the boundary of each S_θ, and dragged along by d_L.

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[23] According to [19] this condition means simply that the deformation δ_{L} and the transformation d_{L} are interchangeable.

A variation of this type leaves the congruence of macroscopical worldlines invariant. If we assume each displaced tube of worldlines to be filled up with the same amount of matter of each kind as the original one, we have $\delta N_r^{\mathrm{d}V} = 0$. Then, putting

$$-N_r^{\mathrm{d}V} \lambda^r = L^{\mathrm{d}V} \quad \ldots \ldots \ldots \quad (42)$$

(41) becomes equivalent with

$$\boxed{\delta \int_{\theta_0}^{\theta_1} d\theta \int_{S_\theta} L^{\mathrm{d}V} = \Big[\int_{S_\theta} P_i^{\mathrm{d}V} \, \delta x^i \Big]_0^{\theta_1}} \quad \ldots \ldots \quad (43)$$

This variational principle shows a great analogy with that of ordinary point mechanics, viz

$$\delta \int_{t_0}^{t_1} L \, dt = [p_i \, \delta x^i]_{t_0}^{t_1}. \quad \ldots \ldots \ldots \quad (44)$$

From (33) and (28) we obtain by Ph. Th. (58), (60) for an *arbitrary* variation δx^h:

$$\delta^0 \int_U \mathfrak{p} \, \mathrm{d}\mathfrak{U} = \int_U (\delta_{\mathrm{L}}^0 \, \mathfrak{p}) \, \mathrm{d}\mathfrak{U} = \int_0^{\theta_1} d\theta \int_{S_\theta} \mathrm{d}\mathfrak{B}_h \, \vartheta^h \, \delta_{\mathrm{L}}^0 \, \mathfrak{p} =$$

$$= \int_0^{\theta_1} d\theta \int_{S_\theta} \mathrm{d}\mathfrak{B}_h \, \vartheta^h \, (\delta_{\mathrm{L}} \, \mathfrak{p} - \mathfrak{p}_r \, \delta_{\mathrm{L}} \, \lambda^r - \mathfrak{p}_i \, \delta_{\mathrm{L}} \, \vartheta^i) =$$

$$= \int_0^{\theta_1} d\theta \int_{S_\theta} (\delta_{\mathrm{L}} \, Z^{\mathrm{d}V} - N_r^{\mathrm{d}V} \, \delta_L \, \lambda^r - P_i^{\mathrm{d}V} \, \delta_{\mathrm{L}} \, \vartheta^i - \mathfrak{Z}^h \, \delta_{\mathrm{L}} \, \mathrm{d}\mathfrak{B}_h),$$

hence

$$\delta^0 \int_U \mathfrak{p} \, \mathrm{d}\mathfrak{U} = \int_0^{\theta_1} d\theta \int_{S_\theta} (\delta_{\mathrm{L}} \, S^{\mathrm{d}V} + \lambda^r \, \delta_{\mathrm{L}} \, N_r^{\mathrm{d}V} + \vartheta^i \, \delta_{\mathrm{L}} \, P_i^{\mathrm{d}V} - \mathfrak{p} \, \vartheta^h \, \delta_{\mathrm{L}} \, \mathrm{d}\mathfrak{B}_h). \quad (45)$$

According to (30) this must vanish for every variation δx^h vanishing at the boundary of U. *The integrand of* (45) *is the quantity which would have to vanish if the operator* δ_L *were replaced by a variation* δ *of the variables* ϑ^h *and* λ^r *alone, without change of the point* x_h (Cf. Ph. Th. (57)).

Hence we have proved:

Theorem 4. The equations of motion (17) *are satisfied if and only if*

$$\boxed{\int_0^{\theta_1} d\theta \int_{S_\theta} (\delta_{\mathrm{L}}\, S^{\mathrm{d}V} + \lambda^r\, \delta_{\mathrm{L}}\, N_r^{\mathrm{d}V} + \vartheta^i\, \delta_{\mathrm{L}}\, P_i^{\mathrm{d}V} - \mathfrak{p}\, \vartheta^h\, \delta_{\mathrm{L}}\, \mathrm{d}\mathfrak{B}_h) = 0} \quad (46)$$

for every variation δx^h vanishing at the boundary of U.

Evidently the last term in the integrand of (46) only defines $\delta_{\mathrm{L}}\, \mathrm{d}\mathfrak{B}_i$. Deforming $\mathrm{d}\mathfrak{B}_i$ simply by "dragging along", we get $\delta_{\mathrm{L}}\, \mathrm{d}\mathfrak{B}_i = 0$, and this term vanishes.

The right member of (30) becomes

$$\int_B P_i^{\mathrm{d}B}\, \delta x^i = \left[\int_{S_\theta} P_i^{\mathrm{d}V}\, \delta x^i \right]_0^{\theta_1} + \int_0^{\theta_1} d\theta \int_{O_\theta} \mathrm{d}\mathfrak{O}_{hj}\, \vartheta^j\, \mathfrak{P}^{h}_{\cdot\, i}\, \delta x^i, \quad . \quad (47)$$

where $\mathrm{d}\mathfrak{O}_{hj}$ are the components of an element of the boundary O_θ of S_θ with appropriate exterior orientation. Hence, if δx^h vanishes at the boundary of each S_θ (though not necessarily over S_0 and S_{θ_1} themselves) and if $\delta_{\mathrm{L}}\, \mathrm{d}\mathfrak{B}_i = 0$, we obtain, equating the right members of (45) and (47):

$$\int_0^{\theta_1} d\theta \int_{S_\theta} (\delta_{\mathrm{L}}\, S^{\mathrm{d}V} + \lambda^r\, \delta_{\mathrm{L}}\, N_r^{\mathrm{d}V} + \vartheta^i\, \delta_{\mathrm{L}}\, P_i^{\mathrm{d}V} - \frac{d}{d\theta}\, P_i^{\mathrm{d}V}\, \delta x^i) = 0.$$

Differentiating with respect to θ_1 and replacing θ_1 by θ we get

$$\int_{S_\theta} (\delta\, S^{\mathrm{d}V} + \lambda^r\, \delta N_r^{\mathrm{d}V} + \vartheta^i\, \delta P_i^{\mathrm{d}V} - \delta x^i\, \frac{d}{d\theta}\, P_i^{\mathrm{d}V}) = 0, \quad . \quad . \quad (48)$$

where the invariant LIE-variation δ_{L} has been replaced by the uninvariant variation δ again.

§ 4. *Variational principles of Fermat's type.*

Let us suppose now that the fluid moves in such a way, that a function $\varphi = \varphi(x^h)$ exists, such that

$$\mathfrak{p}_r\, \partial_i\, \lambda^r = -\, \mathfrak{q}\, \partial_i\, \varphi, \qquad \mathfrak{q} =_{\mathrm{df}} \mathfrak{p}_j\, \vartheta^j. \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Then the equations of motion (25) become:

$$\frac{d_{\mathrm{L}}}{d\theta}\, \mathfrak{p}_i = \mathfrak{q}\, \partial_i\, \varphi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Transvection with ϑ^i shows that the density

$$\mathfrak{a} =_{\mathrm{df}} e^{-\varphi}\, \mathfrak{q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

is a constant of the motion:

$$\frac{d_L \mathfrak{a}}{d\theta} = 0. \qquad (52)$$

Moreover variation of the integral

$$I =_{\mathrm{df}} \int_{\theta_0}^{\theta_1} e^{\varphi}\, d\theta = \int_{\theta_0}^{\theta_1} \mathfrak{a}^{-1}\, \mathfrak{p}_i\, dx^i \qquad (53)$$

with $\delta\theta = 0$ at the boundaries, gives by (50), (51):

$$\delta I = \int_{\theta_0}^{\theta_1} e^{\varphi}\, \delta\varphi\, d\theta = \int_{\theta_0}^{\theta_1} \delta x^i\, d_L\, \mathfrak{a}^{-1}\, \mathfrak{p}_i. \qquad (54)$$

Hence, if the variation is performed by "dragging along" ϑ^h (which is equivalent with the condition that δx^h is dragged along by the operator d_L), i.e. $\delta_L \vartheta^h = -\, d_L \delta x^h / d\theta = 0$, then the integration in the last member of (54) can be carried out by using (52) and we obtain:

Theorem 5[23a]). *If* (49) *is satisfied throughout the fluid, then*

$$\mathfrak{a} =_{\mathrm{df}} e^{-\varphi}\, \mathfrak{p}_i\, \vartheta^i$$

is a constant of the motion and

$$\boxed{\delta \int_{\theta_0}^{\theta_1} e^{\varphi}\, d\theta = \delta \int_{\theta_0}^{\theta_1} \mathfrak{a}^{-1}\, \mathfrak{p}_i\, dx^i = [\mathfrak{a}^{-1}\, \mathfrak{p}_i\, \delta x^i]_{\theta_0}^{\theta_1}} \qquad (55)$$

for any variation δx^h *with* $\delta_L \vartheta^h = 0$[24]).

Corollary: If the fluid is homogeneous and if

$$\partial_i \eta = 0. \qquad (56)$$

throughout the motion, then

$$\delta \int_{\theta_0}^{\theta_1} \pi_i\, dx^i = [\pi_i\, \delta x^i]_{\theta_0}^{\theta_1}. \qquad (57)$$

Proof: By hypothesis $n = 1$. We drop the suffix r and write $\mathfrak{p}'$ for $\partial \mathfrak{p} / \partial \lambda$. Then (56) is equivalent with

$$\partial_i \lambda = -\, \partial_i \pi_j\, \vartheta^j = -\, \pi_j\, \vartheta^j\, \partial_i \log(-\pi_j\, \vartheta^j),$$

23a) The theorem remains valid if to the right member of (49) a term of the form $-\gamma\, \mathfrak{p}_i$ is added and if the first statement is replaced by "$\mathfrak{a}\, e^{\int -\gamma\, d\theta}$ is a constant of the motion", $\mathfrak{a}$ being defined by (51). If to the right members of (50) and (52) a term $\gamma\, \mathfrak{p}_i$ and $\gamma\, \mathfrak{a}$ respectively is added, the demonstration remains valid.

24) In that case the values of θ can be kept constant during the deformation, so that $\delta\theta$ vanishes identically, and that (54) is valid.

where $\pi_i =_{\text{df}} \mathfrak{p}_i/\mathfrak{p}'$, or

$$\mathfrak{p}_r \, \partial_i \, \lambda^r = \mathfrak{p}' \, \partial_i \, \lambda = - \mathfrak{q} \, \partial_i \, log \, (-\pi_j \, \vartheta^j)$$

so that (49) is satisfied with $e^\varphi = -\pi_j \vartheta^j$. Substitution into (52) leads immediately to the proof of the corollary. Evidently (52) holds with $\mathfrak{a} = -\mathfrak{p}'$, i.e. the condition (56) implies that the equation of continuity (cf. § 5 (68)) is satisfied. The equation of continuity, however, is *not* sufficient in order that (56) be satisfied.

As a particular case let us introduce metric again and assume that an equation of state of the form (14) exists with $n = 1$ (homogeneous relativistic fluid) and $f_i = 0$. Then (49) becomes

$$\partial_i \, \lambda = \varepsilon \, \vartheta_0 \, \partial_i \, \varphi \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

with

$$\vartheta_0 = \vartheta^h \, i_h = \sqrt{g_{ij} \, \vartheta^i \, \vartheta^j} = c \, (k \, T_0)^{-1}$$

and

$$\varepsilon = \sqrt{g^{ij} \, \pi_i \, \pi_j} \, , \; \pi_i = - \varepsilon \, i_i .$$

Hence we have also

$$\nabla_i \, \mathfrak{p} = \mathfrak{p}' \, \nabla_i \, \lambda - \mathfrak{p}' \, \varepsilon \, \nabla_i \, \vartheta_0 = \mathfrak{p}' \, \varepsilon \, \vartheta_0 \, \partial_i \, (\varphi - log \, \vartheta_0).$$

A relation $u \, \partial_i \, v = \partial_i \, w$, however, implies that w and u are functions of v alone and that $u = dw/dv$.

As $\mathfrak{p}'$, ε and ϑ_0 are always $\neq 0$ (even > 0), $\varphi - log \, \vartheta_0$ and then also $\mathfrak{p}' \, \varepsilon \, \vartheta_0 = - \mathfrak{p}_i \, \vartheta^i = - \mathfrak{q} = \varrho + \mathfrak{p}$ (cf. (3) and (10)), hence also ϱ is a function of $\mathfrak{p}$ alone[25]) and we have

$$\varphi = log \, \vartheta_0 + \int \frac{d\mathfrak{p}}{\mathfrak{p}' \, \varepsilon \, \vartheta_0} = log \, \vartheta_0 - \int \frac{d\mathfrak{p}}{\mathfrak{p}_i \, \vartheta^i} \, . \quad . \quad . \quad . \quad (59)$$

Or also, introducing the "index-function"[26]) $F =_{\text{df}} e^\varphi/\vartheta_0$, whence

$$log \, F =_{\text{df}} - \int \frac{d\mathfrak{p}}{\mathfrak{p}_i \, \vartheta^i} = \int \frac{d\mathfrak{p}}{\varrho + \mathfrak{p}} , \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

the integral (53) becomes by (49):

$$I = \int_{\theta_0}^{\theta_1} e^\varphi \, d\theta = \int_{\theta_0}^{\theta_1} F \, \vartheta_0 \, d\theta = \int_{\theta_0}^{\theta_1} F \, ds . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

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25) This means that the quantities $\mathfrak{q}$ etc. depend upon the coordinates x^h through $\mathfrak{p}$ alone, i.e. that $\delta \mathfrak{q} = \frac{d\mathfrak{p}}{d\mathfrak{q}} \, \delta \, \mathfrak{p}$ for all variations δ of the form $\delta = \delta \, x^j \, \partial_j$, but of course not for *arbitrary* variations. Relations of this type with respect to the *spatial* coordinates alone are called by BJERKNES barotropic relations. Cf. V. BJERKNES [1] p. 85.

26) Cf. J. L. SYNGE [2] p. 391.

Moreover $\mathfrak{a}^{-1}$ becomes

$$\mathfrak{a}^{-1} = e^{\varphi}\,\mathfrak{q}^{-1} = -F\,\vartheta_0\,(\varrho + \mathfrak{p})^{-1} = -\vartheta_0\,\frac{dF}{d\mathfrak{p}}, \quad \ldots \quad (62)$$

whence

$$\mathfrak{a}^{-1}\,\mathfrak{p}_i = (\mathfrak{a}\,\vartheta_0)^{-1}\,\mathfrak{p}_j\,\vartheta^j\,i_i = F\,i_i.$$

Hence we have proved, the argument being reversible:

Theorem 6. If a homogeneous fluid has an equation of state of the form (14) *and moves in such a way, that ϱ is a function of $\mathfrak{p}$ alone, $k\,T_0\,(dF/d\mathfrak{p})^{-1}$ is a constant of the motion and*

$$\boxed{\delta\int e^{\varphi}\,d\theta = \delta\int F\,ds = [F_i\,\delta x^i]}, \quad \ldots \ldots \quad (63)$$

(*where F is given by* (60), *whereas $F_i =_{\mathrm{df}} F\,i_i$*) *for any variation satisfying the conditions of Theorem* 5.

The variational principle (63), was found in 1924 already by L. P. EISENHART[27]) by a method based on the geometry of paths. It was proved in a different way and discussed further by J. L. SYNGE [2] p. 393. We obtained it here as a metrical specialisation of (54).

Finally we apply theorem 5 to a volume, filled with black radiation, which can be considered as a homogeneous fluid with $\lambda = 0$ and $\mathfrak{q} = \mathfrak{p}_i\,\vartheta^i = -4\,\mathfrak{p}$ (cf. R.G. § 10). Then (49) is automatically satisfied with $\varphi = 0$, so that I simply becomes

$$I = \int d\theta = \int kT\,dt = \int c^{-1}\,k\,T_0\,ds,$$

It is of importance to note that the proper mass of a volume filled with radiation (being greater than the sum of the proper masses of the photons, which is zero) is always positive, viz $3\,\mathfrak{p}\,d\mathfrak{V}/c^2$. Therefore the average velocity of black radiation is always $< c$ (e.g. $= 0$ if the radiation is enclosed in a box at rest). Hence $ds \neq 0$, so that i^h, T_0, etc. remain finite. Of course this does not apply to a single beam of light, which is a limiting case with $\mathfrak{p} = \mathfrak{p}_i\,\vartheta^i = 0$. Moreover (14) is satisfied with $\lambda = 0$. Hence $F = \vartheta_0^{-1} = k\,T_0$. Finally we remark that by theorem 5 $\mathfrak{a} = \mathfrak{q}$, hence $\mathfrak{p}$, hence T_0 is a constant of the motion, so that we have proved

Theorem 7: Black radiation moves always according to the variational principle

$$\delta\int_{\theta_0}^{\theta_1} d\theta = -\tfrac{1}{4}\,[\mathfrak{p}^{-1}\,\mathfrak{p}_i\,\delta x^i]_{\theta_0}^{\theta_1} \quad \ldots \ldots \ldots \quad (64)$$

27) Cf. L. P. EISENHART [1] p. 214—216.

which becomes by metrical specialisation, T_0 being constant along the rays,

$$\delta \int ds = [i_i \, \delta x^i]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Hence the "stream-lines" of black radiation are geodesics.

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Anatomy. — *The cytology of the cortex in the opossum (Didelphys virginiana) considered in relation to some general problems of cortical evolution.* By W. RIESE and G. E. SMYTH [1]. (Communicated by Prof. C. U. ARIËNS KAPPERS.)

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In the extensive literature devoted to the comparative anatomy of the cerebral cortex it would appear that in the cytotectonic descriptions too little attention is payed to the differentiation of the constituent cells themselves. So the cortical lamination of the opossum has been carefully studied by GRAY (1924); the excitable areas of the cortex of this animal by GRAY and TURNER (1924) and the origin and course of the pyramidal tracts by TURNER (1924). Nevertheless little or no information is available on the structural differentiation of the cells in the several cortical laminae and in representative cortical areas.

In the present paper we shall deal with some cellular differentiations in the cortex of the opossum as an addition to our recent observations on the cellular structure of the thalamus and corpus striatum and other subcortical centers of this primitive mammal [2].

In consequence of the lack of systematic phylogenetic studies of cellular differentiation in mammals no absolute standard exists whereby the degree of differentiation of nerve cells may be tested. In these observations the criteria adopted are based on ontogenetic studies, in particular on the conclusions arrived at by RIESE (1939) from investigations on the development of the brain of the bear (Ursus arctos). They are in close agreement with some earlier observations on the development of cortical cells in pig and rabbit embryos reported by PATON (1900).

Realizing the fundamental difference between embryonic and adult cell structures, also in lower mammals, the comparisons established in this paper are only meant to indicate certain analogies. With this reservation the criteria of structural differentiation may be summarized as follows:

1. The amount of cytoplasm, particularly in relation to the bulk of the nucleus, and the sharpness of demarcation between cytoplasm and the surrounding substances. The more primitive the cell, the more poorly demarcated seems the cell outline. The bulk of the cytoplasm relative to that of the nucleus tends to increase with the progressive evolution of the cell.

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[2]) These observations will be communicated soon.

2. The intensity with which the cytoplasm is stained by basic solutions both absolutely and relatively to the nucleus. This is a measure of the amount of chromophil substance present in the cell since the diminution of the chromophil substance in the nucleus and its increase in the cytoplasm run parallel with the maturation of the cell.

3. The presence of distinct chromophil granules in the cytoplasm. It seems that their presence indicates a high degree of differentiation.

4. The form of the nucleus and, more particularly, the number of nucleoli and their position within the nucleus. The presence of numerous nucleoli, more especially when disposed peripherally is caracteristic of a primitive type of cell.

5. Finally the position of the nucleus in the cell, central or excentric, and the degree of development of cell processes appear to be of some significance in conjunction with the criteria already enumerated.

Material and methods. The opossum (Didelphys virginiana) on which the investigations were made was a young adult male of good nutrition which had died suddenly from an acute cardiac affection.

The brain, after fixation in 10 % formol, was embedded in celloidin and sectioned serially in the vertical plane. Alternate sections were stained by the methods of NISSL (Thionin blue) and SPIELMEYER.

There is little difficulty in identifying the principal cortical areas with those described by GRAY and TURNER (l.c.). The degree of structural differentiation of the nerve cells of the various laminae was examined in the following areas:

1. Area praepiriformis — Palaeo-cortex (KAPPERS)
2. Cornu ammonis — Archi-cortex (KAPPERS)
3. Area praeorbitalis — Neo-cortex
4. Area parietalis — Neo-cortex
5. Area striata — Neo-cortex

No attempt has been made to study the cellular structure in each cortical area. The study of some clearly defined and representative areas would best serve the purpose of this investigation, which is intended only to find some principles indicative of the general line along which the cortex develops.

1. *Area praepiriformis.* This cortex is composed of three laminae of which the 1st and 3rd are deep and the 2nd is narrow. The 1st lamina is very poor in cells and such as occur resemble in every way those of the second lamina.

In the 2nd lamina the cells are extremely numerous, medium sized, polygonal and seem to be scantly provided with processes. The cytoplasm stains deeply and diffusely. The nucleus stains less deeply than the cytoplasm and contains, as a rule, one large and prominent nucleolus.

In the 3rd lamina the cells are less dense but still very numerous. They are somewhat larger than those in L. 2 and their size increases slightly

in those lying nearer the white matter. Many pyramidal forms are seen, but the majority is polygonal.

Each cell has several prominent processes. The cytoplasm is distinctly paler than in L 2, and contains numerous small chromophil granules. The nucleus is large and round, usually excentric in position and provided with a distinct nuclear membrane. It stains lightly and possesses a single darkly staining nucleolus and a few threads of chromophil substance. The largest cells located near the white matter, often show a vacuolated cytoplasm. A considerable glial satellitosis is frequently seen.

Apparently in this most primitive cortical area the individual cells are very mature. It is significant that the cells in L 3 are somewhat more highly differentiated than in L 2 and that, in fact, the degree of differentiation shows a progressive increase towards the white matter.

2. *Cornu ammonis.* (Archicortex (KAPPERS), Cortex totoparietinus bistratificatus holoprotoptychos (ROSE), cortex heterogeneticus rudimentarius (BRODMANN).)

The cells of the ammonal area vary in size from that of medium to rather large cells. The majority are pyramidal but many ovoid and many pear shaped forms occur. The cytoplasm is scanty in relation to the size of the nucleus and contains numerous coarse fragments of chromophil substance. The nucleus is round, pale and possesses, as a rule, *one* deeply staining nucleolus. The cell processes are well developed: many are directed into the lamina zonalis in a vertical and parallel sense. The description agrees with that given by ROSE (1927) for Didelphys azarae. The cells of the *fascia dentata*, generally described as "granular" cells (ROSE, l.c.), only show a narrow but clearly visible ring of cytoplasm showing a light pink stain. The nucleus is round and contains numerous granules of dark chromophil substance located peripherally. These cells resemble those of the habenular ganglion of this animal. Apparently the cells of the cornu ammonis are highly differentiated, showing many of the characters of mature ganglion cells, while those of the fascia dentata are of very primitive type.

Area praeorbitalis. The lamination in this area is very distinct. The 4th lamina (l. granularis interna) is lacking i.e. the cortex is agranular. The 1st lamina (l. marginalis) is broad but only contains a few scattered cells towards its deeper part. In the 2nd lamina (l. granularis externa) the cells are very densely packed. They are mostly of triangular shape with the apex directed towards the surface. These cells stain so deeply that their internal structure cannot be defined. Each cell possesses several long and slender processes. The 3rd lamina (l. pyramidalis) is also broad and contains many cells. These cells are larger than in L 2, polygonal and possess rather short processes. The cytoplasm is abundant and stains lightly. No NISSL granules are seen. The nuclei are large, round and stain

more deeply than the cytoplasm. Each of them contains a net-work of chromophil substance and one or two dark nucleoli. In the 5th lamina (l. ganglionaris) the cells are generally pyramid shaped, their orientation being such that the apex is directed toward the white matter. It is to be noted that by their medium size these cells are very different from the giant cells of the motor cortex of higher mammals. The outline of the cell is well delimited and the processes are long. The cytoplasm is rich in tigroid granules only their axon hillock is devoid of these granules. The nucleus, located centrally, is pale and contains one deeply stained and centrally placed nucleolus. These cells are relatively scanty in number. The 6th lamina (l. multiformis) is poorely delimited towards the white matter. The cells are rather scanty. They stain deeply but nevertheless their boundaries are rather ill defined. They are polygonal, somewhat smaller than those of the 3rd and 5th laminae and their processes are only seen with difficulty. Their nuclei, as far as may be observed, only contain one dark nucleolus.

The cytotectonic arrangement in this area as well as the differentiation of the cells themselves evidently are highly evolved. As in the areas previously described their evolution progresses from without inwards.

Area parietalis. As in the other areas only an occasional cell is found in the deeper part of the 1st lamina. In the 2nd lamina the cells are closely packed to form rather narrow zones. They are polymorphic, poorly defined and each possesses one or more short processes. The cytoplasm stains so deeply and uniformly that only occasionally is it possible to see the still more deeply stained nuclei with their numerous nucleoli. In the 3rd lamina the cells are pyramidal in form and sharply outlined. The cytoplasm stains with moderate intensity and NISSL granules are absent. The nucleus is relatively large, excentric in position. It usually contains many nucleoli, but occasionally one nucleolus stands out more prominently than the others. From the top of the cells a big process arises directed towards the lamina zonalis. The 4th lamina is difficult to define, its cells resembling closely those in the 3rd lamina. Generally, they are round or ovoid in outline. The cytoplasm is much less deeply stained than in the cells in the external granular layer but there is no trace of granules. The nucleus accounts for the chief bulk of the cell; it stains more deeply than the cytoplasm and contains numerous chromophil granules which show a tendency to be grouped at the periphery. Short processes are seen, but only occasionally. The 5th lamina is a distinct layer of medium sized nerve cells showing a linear arrangement of several layers. The cells are sharply defined and have one thick apical process and finer lateral and basal processes. The cytoplasm is deeply but irregularly stained as if from the presence of coarse masses of chromophil substance. The nucleus stains less deeply than the cytoplasm and contains a solitary dark nucleolus. The

6th lamina merges with the 5th but its cells are more scanty and their orientation is more irregular. They vary in size and, in general, are very deeply staining. In those cells in which the nucleus is seen, it is found to contain one nucleolus. A moderate glial satellitosis is of common occurrence in this layer.

In this area cytoarchitectonic differentiation is in advance of the differentiation of cellular structure. In the 5th lamina the nerve cells are less highly evolved than in the corresponding lamina of the area praeorbitalis. It is noteworthy that a considerable difference exists between the structure of the cells in the two granular layers.

Area striata. The 1st lamina, as in the other areas examined, only contains scattered nerve cells. Such as occur are small and so intensely stained that their internal differentiation cannot be distinguished. The cells in the 2nd lamina form a narrow but closely packed zone and their staining property is identical with those in the 1st. They are provided with short processes, which gives them a spiderlike appearance. The 3rd lamina is composed of medium sized pyramidal cells arranged in parallel rows. The cytoplasm is deeply and uniformly stained but the nucleus is nearly always visible. Their nuclei are to intensely stained to ascertain the internal structure. Each cell has two or more rather slender processes. The 4th lamina is deep and very distinct from the adjacent laminae. It is composed of medium sized fusiform cells, a trifle larger than those of the 2nd lamina. In some instances slender spider-like processes are seen. The cytoplasm stains far less deeply than that in the 2nd layer and with a slight pink colour only. Distinct NISSL granules cannot be observed in it. The nucleus is relatively larger, generally excentrically placed, and more deeply stained than the cytoplasm. Each contains numerous chromophil granules of which two or more stand out prominently. In the 4th layer of the area striata of the opossum there is no trace of sublamination such as is found in higher mammals. Although in low magnification a distinction can be made between the 5th and 6th laminae, it is impossible to do so on the basis of a difference of cellular structure. The cells in these layers are more widely dispersed than in L 4 and their size tends to be somewhat larger. The cytoplasm stains less uniformly but in all other respects these cells are hardly to be differentiated from those of the 4th lamina. A certain richness in glial elements exists in the 6th lamina and a marked satellitosis is not uncommon. This phenomenon is most evident and most frequent around displaced ganglion cells which are to be seen almost as deep as the ependyma of the lateral ventricles.

The most striking feature of this area is the primitive structure of its cells, strongly contrasting with its well differentiated cytoarchitecture (orientation and degree of density of the cells). Evidently the cells in the deeper laminae only are more highly organized.

Discussion.

From our observations it appears that, although an equal degree of laminar differentiation is not present throughout the whole cortex, this *laminar differentiation is in advance of the degree of cellular differentiation in the areas examined.* In some areas the disparity is very great. In all the areas examined *the most highly differentiated cells are constantly found in the 5th layer.*

These observations agree with the fact observed in the ontogenetic development of the cortex in which the formation of the cytoarchitectural pattern always preceeds the differentiation of the constitutive cells, and with the fact that at a period when the cells composing the lamina granularis primaria still are in the neuroblastic stage, certain deeper cells, separated from it by a zone poor in cells, already possess the characteristic form of ganglion cells. In this layer, the primitive 5th cortical lamina, cellular differentiation begins. Nothwithstanding these analogies between the mature state of the cortex of a primitive mammal and certain embryonic stages in higher mammals these analogies do not, of course, mean that in both cases the same causal factors are in operation.

A comparison between the degree of cellular differentiation in the cortical areas examined shows an additional interest viz. that a particular degree of cellular differentiation is characteristic of the palaeocortex, archicortex and neocortex. Thus, *the cells of the area praepiriformis (palaeocortex) and those of Ammon's horn (archicortex) are very highly differentiated, whereas the cells of the neocortex, with the exception of the area praeorbitalis, are in general poorly differentiated.*

It follows that the subdivision of the cortex into palaeo- and archicortex on one hand and neocortex on the other corresponds with the degree of cellular evolution: indeed the constitutive cells of the recent areas are less highly organized than those of the ancient and more primitive areas.

Before concluding this discussion two observations of minor interest seem worthy of brief mention.

If the internal structure of the cells composing the second and fourth laminae are compared, it will be found that they show considerable points of difference. In all the areas examined it is true to say that the cells of the internal granular lamina have a slightly more advanced structure than those of the external granular lamina. But what chiefly distinguishes them from the "granular" cells of higher mammals is the possession of a particularly developed cytoplasmic cell body.

Finally it seems worth while to draw attention to the phenomenon of satellitosis as observed in the brain of the opossum. When present this phenomenon was confined to the deeper laminae and it reached its maximum development around the aberrant nerve cells scattered in the white matter towards the ependyma of the ventricle. Although the present

observations afford no clue to its significance this peculiar limitation should be mentioned.

Summary and conclusions.

1. The cytology of the cerebral cortex of the opossum (Didelphys virginiana) has been examined in NISSL preparations and the findings are discussed in relation to some of the general problems of the evolution of the cortex.

2. In all the areas examined cytoarchitectural differentiation is in advance of cellular differentiation. The most highly evolved cells are found in the area praepiriformis and cornu ammonis; the least differentiated are those of the neo-encephalon. The area praeorbitalis is however exceptional in that the cells of the 5th lamina are highly differentiated. Irrespective of the area the cells of the 5th lamina are invariably the most highly differentiated. Both these observations are closely in accord with the facts of ontogenetic development.

3. In general, the cortical areas considered as primitive possess the most highly evolved cellular elements, whilst the more recent areas are composed of cells showing a primitive organization.

4. The cells in the 2nd and 4th laminae are not identical in structure but both possess an easily visible cytoplasmic body.

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